Group Laplacians

Cayley Graphs

Definition (Cayley graph)

Let $G = \langle \mathcal{S} | \mathcal{R} \rangle$ be a (finitely presented) group. Cayley graph $\mathsf{Cay}(G, \mathcal{S})$ is a graph (V, E), where

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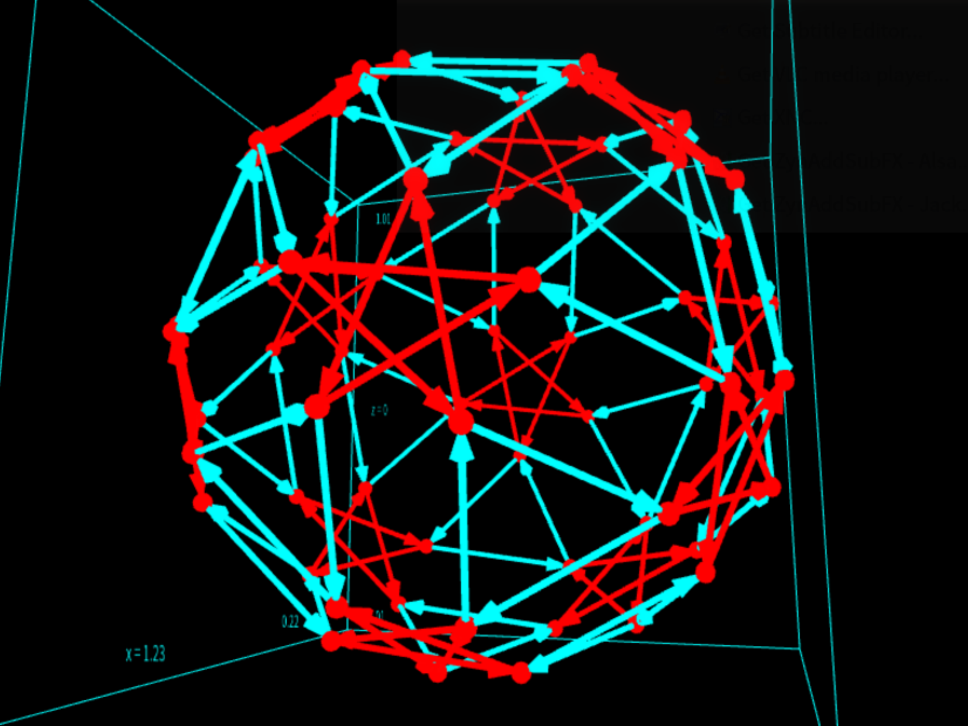
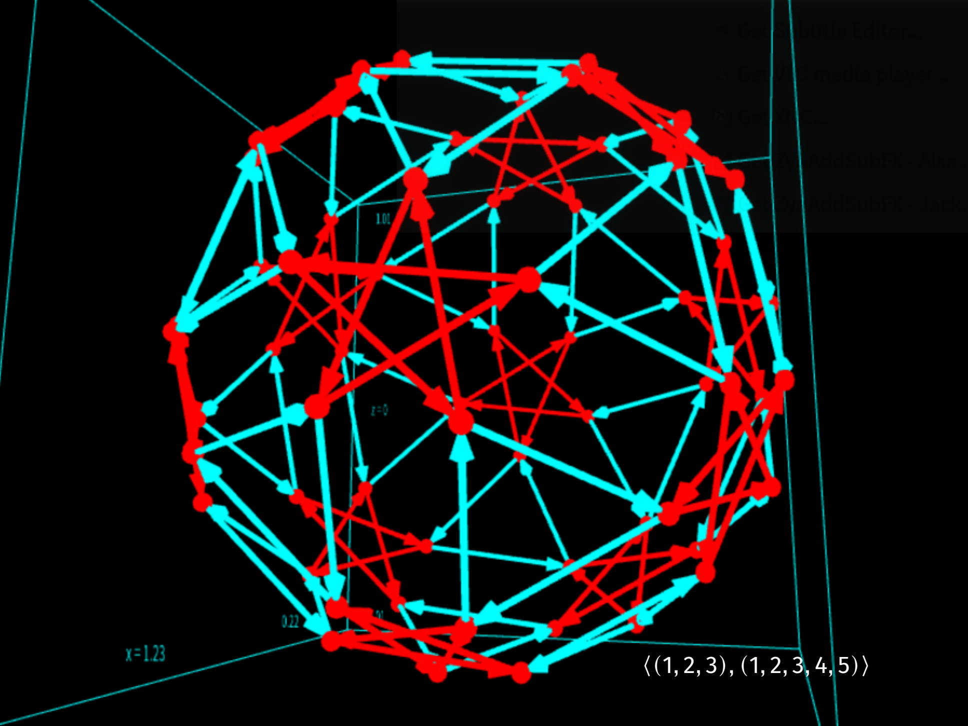
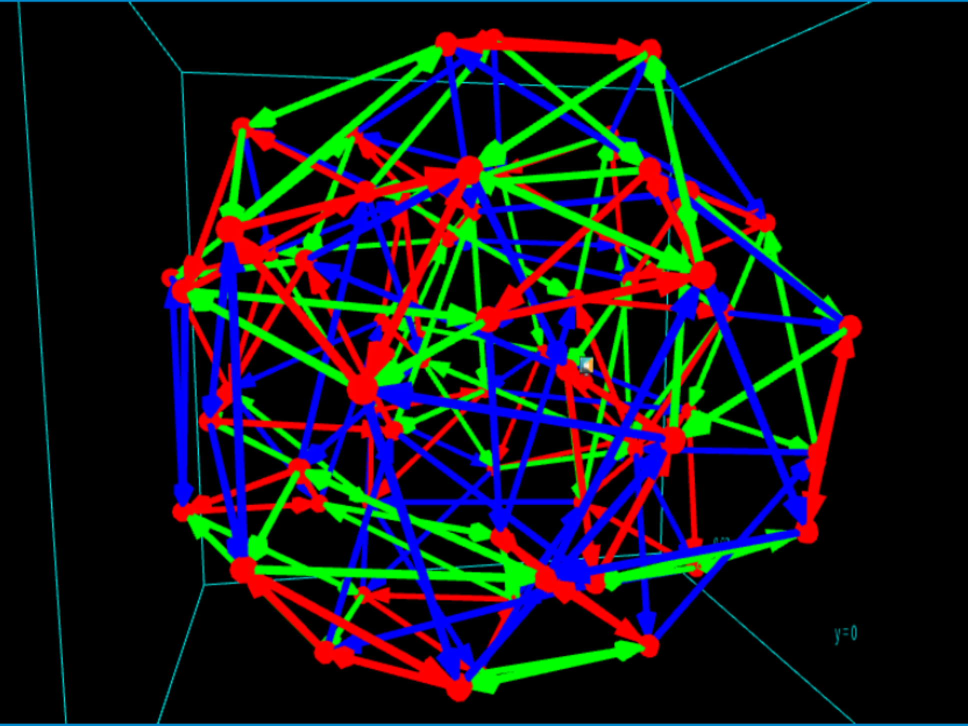
- $\mathsf{Cay}(G, \mathcal{S})$ are very symmetric: links of all vertices are isomorphic.


1.01
z=0
0.22
x = 1.23



$\langle (1,2,3), (1,2,3,4,5) \rangle$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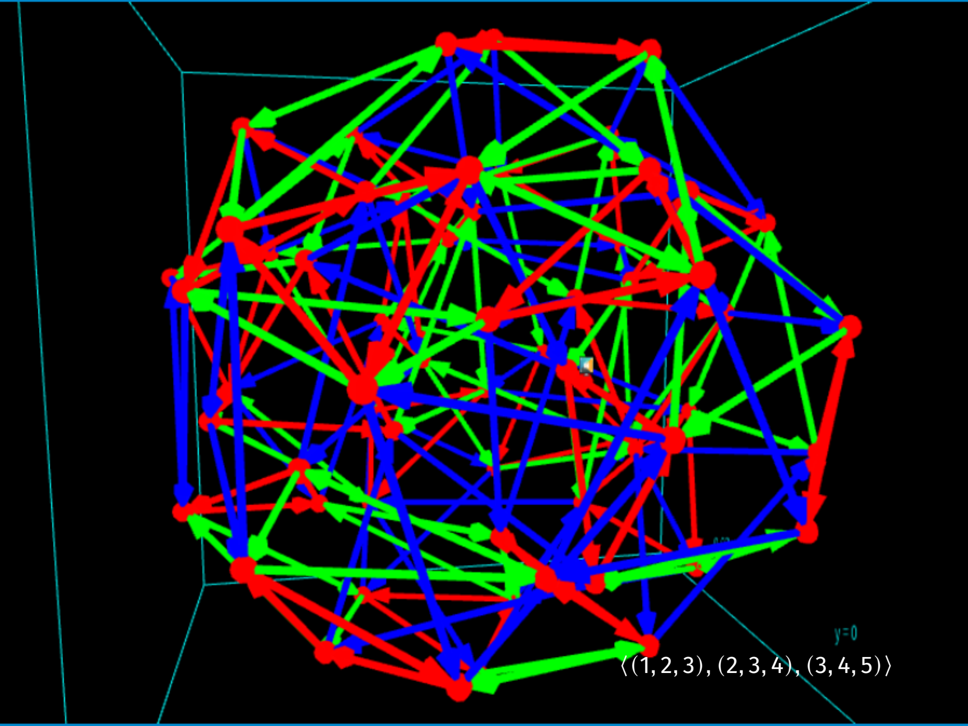

$\langle(1,2,3),(2,3,4),(3,4,5)\rangle$

Group algebras

Convention: $G = \langle S \mid R \rangle$ is a finitely presented group generated by a fixed **symmetric generating set** (i.e. $S^{-1} = S$).

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- the involution $*\colon \mathbb{R}[G] \to \mathbb{R}[G]$ induced by $g \mapsto g^{-1}$ and trivial on $\mathbb{R}$ gives $\mathbb{R}[G]$ the structure of $*$-algebra, e.g.

$$(1e - 2g + 3g^{-1}h^2)^* = 1e - 2g^{-1} + 3h^{-2}g.$$

Definition

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Group Laplacian

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- spectrum of Δ is real and non-negative;
- the second eigenvalue λ_1 is called the **spectral gap**

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots$$

Property (T)

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- For a unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ of G on a Hilbert space denote by
$$\mathcal{H}^\pi = \{v \in \mathcal{H} : \pi(g)v = v \text{ for all } g \in G\}$$
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- We define
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Property (T) – meaningless gist

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- Property (T) is a **hard analytic property**;
- the constant $\kappa(G, S) \geqslant 0$ is a quantitative indicator of the property;
- the exact value of κ does depend on S; its positivity does not;
- estimating the constant is very hard;

Property (T) – examples

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Theorem (Lubotzky & Pak, 2000)

Let K be a finite group generated by $k \leq n$ elements. If $\mathrm{SAut}(F_n)$ has property (T) with constant $\kappa = \kappa(\mathrm{SAut}(F_n), \{\mathrm{transvections}\}) > 0$, then ***PRA walk*** *has fast mixing time,* Random group elements in finite groups: estimating mixing time of the **Product Replacement Algorithm** depends on the Kazhdan's constant of SAut(F_n), the special authomorphism group of the free group:

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Note

We do observe fast mixing time in practice for large n.

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Corollary

We have $\lambda \in [0, \lambda_1) \iff \Delta^2 - \lambda\Delta \geqslant 0$, i.e. if there exists $\lambda > 0$ such that $\Delta^2 - \lambda\Delta \geqslant 0$, then G has property (T) with

$$\sqrt{\frac{2\lambda}{|S|}} \leqslant \kappa(G, S).$$

How to prove that $\Delta^2 - \lambda\Delta \geqslant 0$?

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Easy to check refutation (find an $x \in \mathbb{R}^n$ such that $f(x) < 0$).

Does there exist a *witness* for confirmation that is also easy to verify?

Positivity

- **analytic**: the value $f(x) \geqslant 0$ for every $x \in \mathbb{R}^n$
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We say that f admits a sum of squares decomposition when

$$f = \sum_i f_i^2 \quad \text{for some } f_i \in \mathbb{R}[\mathbf{x}].$$

Hilbert's 17th problem

Theorem (Hilbert, 1888)

An everywhere non-negative polynomial $p \in \Sigma^2\mathbb{R}[x_1, \ldots, x_n]$ *(is a sum of squares) if and only if either*

- $n = 1$ *(univariate polynomial), or*
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Example

$\left(x^2 + y^2 + 1\right)\left(x^4y^2 + x^2y^4 - 3x^2y^2 + 1\right)$ is a sum of squares!

Analytic and algebraic positivity

- $p \in \mathbb{R}[\mathbf{x}]$ is *positive* iff $p(t) \geqslant 0$ for all $t \in \mathbb{R}^n \implies$ **analytic positivity**.
- $p \in \mathbb{R}[\mathbf{x}]$ is *positive* iff p is a sum of squares (of rational functions) $\implies q^2 p \in \Sigma^2 \mathbb{R}[\mathbf{x}] \implies$ **algebraic positivity**.

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Problem

How to find such sum of squares (SOS) decomposition?

SOS decompositions

We can write f as a **quadratic function of monomials**.

Example

$$f = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$$

can be written as

$$f = \left[x^2, xy, y^2\right] \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7+2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} = \mathbf{x}^T P(\lambda)\mathbf{x}$$

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P is so called **Gramm matrix** for f.

Quadratic function of monomials

Lemma

f admits a sum of squares decomposition iff there exists a ***positive semidefinite*** *Gramm matrix for some (sub)basis* $\mathbf{x}$ *of* $\mathbb{R}[x_1, \ldots x_n]$.

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If P is positive semidefinite, then $P = Q^T Q$ (e.g. by Cholesky) and

$$f = \mathbf{x}^T P \mathbf{x} = \mathbf{x}^T Q^T \cdot Q \mathbf{x} = (Q\mathbf{x})^T \cdot (Q\mathbf{x}).$$

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Example

For example for $\lambda = 6$ we have

$$P(\lambda) = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} = Q^T \cdot Q \quad \text{for } Q = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

Therefore f admits a SOS decomposition

$$f = \left(2xy + y^2\right)^2 + \left(2x^2 + xy - 3y^2\right)^2.$$

Linear programming:

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Semi-definite programming

- optimise linear functional
- on a polytope intersected with the cone of PSD matrices (spectrahedron)
- weak duality, non-unique solutions
- even feasibility is a hard problem!

PSD problem formulation

optimisation variables: a, b_1, b_2, c, λ.

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Example (PSD problem)

$$
\begin{aligned}
\text{maximise:} \quad & \lambda \\
\text{subject to:} \quad & \lambda \geqslant 0 \\
& c = 1 - \lambda \\
& b_1 + b_2 = 4 \\
& a = 2 \\
& \begin{bmatrix} c & b_2 \\ b_1 & a \end{bmatrix} \geqslant 0
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tries to maximise λ as long as $(2x^2 + 4x + 1) - \lambda \geqslant 0$.

Semidefinite program

$$\begin{aligned} \min_{P \in \mathcal{S}^N} \quad & \langle C, P \rangle \\ \text{subject to:} \quad & \langle A_i, P \rangle = b_i, \quad i = 1, 2, \ldots, m \\ & P \succeq 0. \end{aligned}$$

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Is $f(x) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$ always non-negative? $\rightarrow$ sum of squares relaxation: Does there exist psd matrix P s.t.

$$f = \mathbf{x}^T P \mathbf{x} = \langle \mathbf{x}\mathbf{x}^T, P \rangle?$$

Feasibility problem with

$$X = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}, XX^\top = \begin{bmatrix} x^4 & x^3y & x^2y^2 \\ \cdot & x^2y^2 & xy^3 \\ \cdot & \cdot & y^4 \end{bmatrix}, A_{x^4} = \begin{bmatrix} 1 & 0 & 0 \\ \cdot & 0 & 0 \\ \cdot & \cdot & 0 \end{bmatrix}, A_{x^3y} = \begin{bmatrix} 0 & 1 & 0 \\ \cdot & 0 & 0 \\ \cdot & \cdot & 0 \end{bmatrix}, \text{ etc.}$$

$b = (b_i)$ is the vector of coefficients of f.

NC positivity: $*$-algebras

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- $\mathbb{R}[G]$, $g \mapsto g^{-1}$ + linear extension (real group algebra).

NC sums of squares

Definition

If $\mathcal{A}$ is a $*$-algebra, then the cone of sum of squares is defined as

$$\Sigma^2\mathcal{A} = \left\{ \sum_{i=1}^{n} \xi_i^* \xi_i : \quad \xi_i \in \mathcal{A}, \quad n \in \mathbb{N} \right\}.$$

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$$\Delta = |S|e - \sum_{g\in S} g = \frac{1}{2}\left(2|S|e - \sum_{g\in S} g^* + g \right) = \frac{1}{2}\sum_{g\in S}(2e - g^* - g)$$
$$= \frac{1}{2}\sum_{g\in S}(1-g)^*(1-g)$$

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Example (Free polynomial algebra)

For a $*$-invariant element $a = \sum_g a_g g \in \mathbb{R}\langle x_1, \ldots, x_d \rangle$ we write $a \succcurlyeq 0$ if for all $n \in \mathbb{N}$ and for every choice of matrices $A = (A_1, \ldots, A_d)$ (each $A_i \in M_n(\mathbb{R})$), the homomorphism φ_A defined by

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- This corresponds directly to evaluation of polynomials: the only representations of polynomial rings are

$$\varphi_t(p) = p(t) \quad \text{for } t \text{ in } \mathbb{R}^d.$$

NC-Positivstellensatz

Theorem (Abstract Positivstellensatz: K. Schmüdgen, ...)

For a $*$-invariant element $a \in \mathcal{A}$ the following conditions are equivalent.

1. $a \geqslant 0$ *(with respect to all $*$-representations of $\mathcal{A}$),*
2. $a + \varepsilon u \in \Sigma^2\mathcal{A}$ *for all $\varepsilon > 0$, where u is an interior point of $\Sigma^2\mathcal{A}$.*

Recall that for a group G:

- **property (T) is hard analytic;**
- **but (T) reduces to the positivity of $\Delta^2 - \lambda\Delta$ in $\mathbb{R}[G]$**
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(as long as we know some interior points u)

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$\mathbf{1e}$ is an interior point of $\Sigma^2\mathbb{R}[G]$, i.e.

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This of no use for us: SOS decompositions $\Delta^2 - \lambda\Delta + \varepsilon\mathbf{e} = \sum \xi_i^*\xi_i$ may be very diffrent for different ε.

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for all ε simultanuously!

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(Modulo certification of the result.)

A few numbers

G	n	m	λ	$\|r\|_1 <$
$\mathrm{SL}(3,\mathbb{Z})$	390,287	935,021	0.5405	$5.2 \cdot 10^{-7}$
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(There must be a better way!)

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Block-diagonalization

Represent P as block diagonal direct sum of psd matrices:

1. Chordal decomposition (exploit sparsity pattern in A_is)
2. Wedderburn(-Artin) decomposition for matrix algebras
   (group symmetry, general *-algebras, Jordan algebras, …)

Group symmetry

Invariant problems

Optimization problem

$$\begin{aligned} \text{maximise:} \quad & \langle c^T, P \rangle \\ \text{subject to:} \quad & P \succcurlyeq 0 \\ & \langle A_i, P \rangle \geqslant b_i, \quad i = 1, \ldots, m. \end{aligned}$$

- G is a finite group
- G acts linearly, orthogonaly on $\mathbb{R}^n$ (space of variables)
- G acts linearly, orthogonaly on $\mathbb{R}^m$ (space of constraints)

Group symmetry invariance: Example

Robinson form

$$R(x,y) = x^6 + y^6 - x^4y^2 - y^4x^2 - x^4 - y^4 + 3x^2y^2 - x^2 - y^2 + 1.$$

R is invariant under the following operations **on monomials**

$$\alpha_1 : (x,y) \mapsto (y,x)$$
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- the symmetry of monomials leads to the symmetry of the psd matrix P.
- In this case: invariant problem = defining polynomial is invariant!

Misleading quote of the day

*The **structure of simplifications** that can be derived from group symmetry does **not depend** on the optimization problem.*

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Lemma (Schur)

Suppose that M, P *be two linear* $\mathbf{G}$*-maps,* $M = \mathbf{m}_1 \oplus \mathbf{m}_2$ *for two irreducible projections* $\mathbf{m}_i$ *and such that* $MPM^{-1} = P$*. Then*

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- a $\boldsymbol{G}$-invariant subspace $\boldsymbol{V}$ is **irreducible** if its only $\boldsymbol{G}$-invariant subspaces are $\{0\}$ and $\boldsymbol{V}$,
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Miracle: When $M = M_g$ are given by a linear G-action they can be simultaneously diagonalized to isotypical blocks!

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P is invariant under group symmetries $\iff M_g P M_g^{-1} = P$ for every $g \in G \implies$ P admits a block-diagonal structure.

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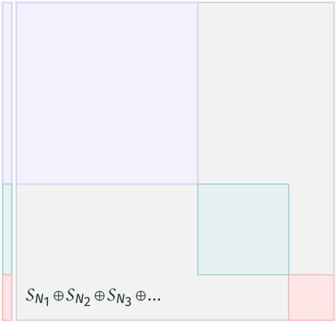

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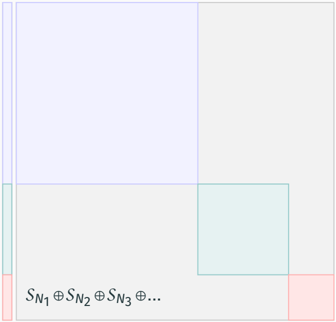

These projections live in the group *-algebra in a basis-free form!

Example: `SymbolicWedderburn.jl`

```
using PermutationGroups, DynamicPolynomials
using SymbolicWedderburn
G = PermGroup([perm"(1,2)", perm"(1,2,3,4)"]) # Sym(4)
@polyvar x[1:4];  basis = monomials(x, 0:2) # 15 monomials
symmetry_adapted_basis(Rational{Int}, G, VariablePermutation(), basis,
    semisimple=true)
```

Isotypical/**semisimple** blocks when acting on `basis`:

$$B_1 = \begin{bmatrix} 1 \\ x_1 + x_2 + x_3 + x_4 \\ x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 \end{bmatrix} \quad B_2 = \begin{bmatrix} x_1 - x_4 \\ x_2 - x_4 \\ x_3 - x_4 \\ x_2^2 - x_4^2 \\ x_3^2 - x_4^2 \\ x_1^2 - x_4^2 \\ x_1x_2 - x_3x_4 \\ x_1x_3 - x_2x_4 \\ x_1x_4 - x_2x_3 \end{bmatrix} \quad B_3 = \begin{bmatrix} x_1x_2 - x_1x_4 - x_2x_3 + x_3x_4 \\ x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 \end{bmatrix}$$

We went from 15×15-psd constrain to sizes $(4 \times 4, 9 \times 9, 2 \times 2)$.

Group algebra and projections

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Group algebra $\mathbb{R}[G]$

- elements of $\mathbb{R}[G]$ are (finitely supported) functions $a \colon G \to \mathbb{R}$, usually written as $a = \sum_g a_g g$

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Projections onto isotypical subspaces live in $\mathbb{R}[G]$ in a matrix-free form.

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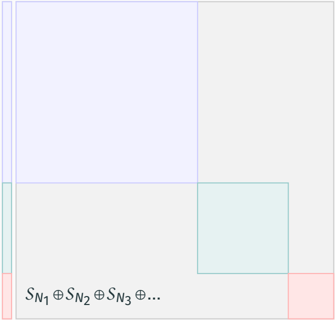
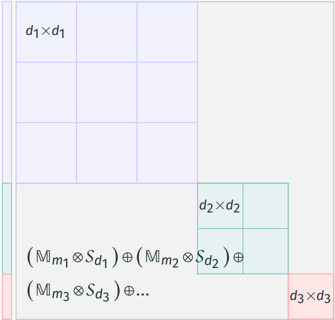
Fact:

Projections onto isotypical subspaces live in $\mathbb{R}[G]$ in a matrix-free form.

$$B_3 = \begin{bmatrix} x_1x_2 - x_1x_4 - x_2x_3 + x_3x_4 \\ x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 \end{bmatrix} \longleftrightarrow p_3 = \frac{1}{12}\begin{pmatrix} 2() - (2,4,3) - (2,3,4) + 2(1,2)(3,4) - \\ (1,3,2) - (1,4,2) - (1,4,3) + (1,3)(2,4) - \\ (1,2,3) + 2(1,4)(2,3) - (1,2,4) - (1,3,4) \end{pmatrix}$$


$S_{N_1} \oplus S_{N_2} \oplus S_{N_3} \oplus \ldots$



$d_1 \times d_1$
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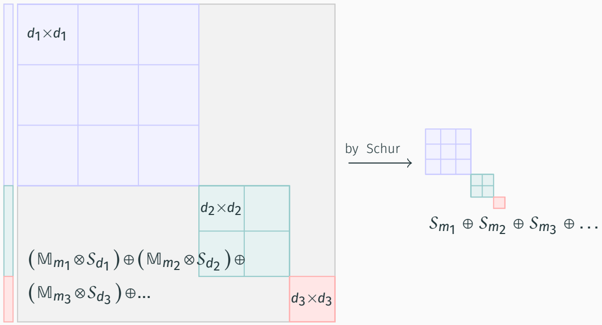

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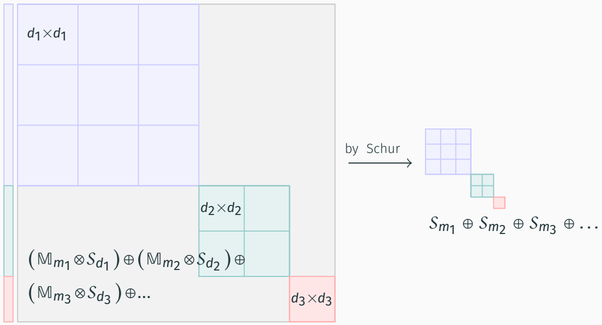
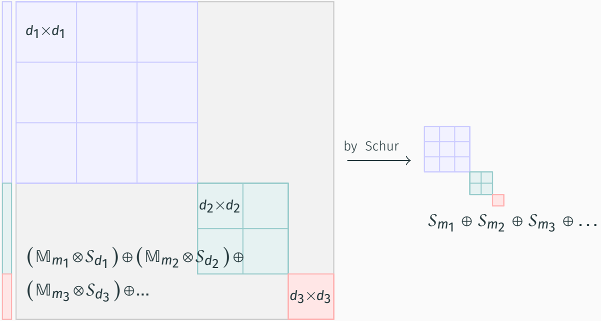


$$B_3' = \frac{1}{2}\begin{pmatrix} x_1x_2 - x_1x_4 - x_2x_3 + x_3x_4 + \\ x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 \end{pmatrix} \longleftrightarrow q_3 \cdot p_3 \quad \text{where } q_3 = \frac{1}{2}\left(() + (3,4)\right)$$



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Open problem

Given an isotypical projection $p \in \mathbb{R}[G]$ how to find a projection $q \in \mathbb{R}[G]$ so that $p(q) = 1$?

Example: SymbolicWedderburn.jl

```
# [ ... ]
symmetry_adapted_basis(Rational{Int}, G, VariablePermutation(), basis
  [, semisimple=false])
```

Simple blocks when acting on `basis`:

$$B_1' = \begin{bmatrix} 1 \\ x_1 + x_2 + x_3 + x_4 \\ x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 \end{bmatrix} \quad B_2' = \begin{bmatrix} \frac{1}{3}(3x_1 - x_2 - x_3 - x_4) \\ \frac{1}{3}(3x_1^2 - x_2^2 - x_3^2 - x_4^2) \\ x_1x_2 + x_1x_3 + x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4 \end{bmatrix}$$

$$B_3' = \left[\frac{1}{2}(2x_1x_2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 + 2x_3x_4)\right]$$

Reduction: $15 \times 15 \to (4 \times 4, 9 \times 9, 2 \times 2) \to (4 \times 4, 3 \times 3, 1 \times 1)$-psd constraints.

Large scale example

Optimization problem from geometric group theory[1]:

Estimate the spectral gap of the group Laplacian for $\mathrm{Aut}(F_5)$

If $\Delta^2 - \lambda\Delta \geqslant \mathbf{0}$ then $(0, \lambda)$ is not in the spectrum.

[1] Kaluba, M., Nowak, P.W. & Ozawa, N. $\mathrm{Aut}(F_5)$ has property (T). *Math. Ann.* **375**, 1169–1191 (2019). https://doi.org/10.1007/s00208-019-01874-9

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- relax $\Delta^2 - \lambda\Delta \geqslant 0$ as sum of squares problem:
- psd-constraint of size $4\,641 \times 4\,641$, $1.1 \cdot 10^7$ constraints
- symmetry group: $S_2 \wr S_5$ (3840 elements)
- After symmetrization:
  - 29-blocks (largest: 58×58) ($13\,232$ variables in total)
  - $7\,230$ constraints
- Solvable in 20 minutes to $\varepsilon \sim 10^{-12}$!

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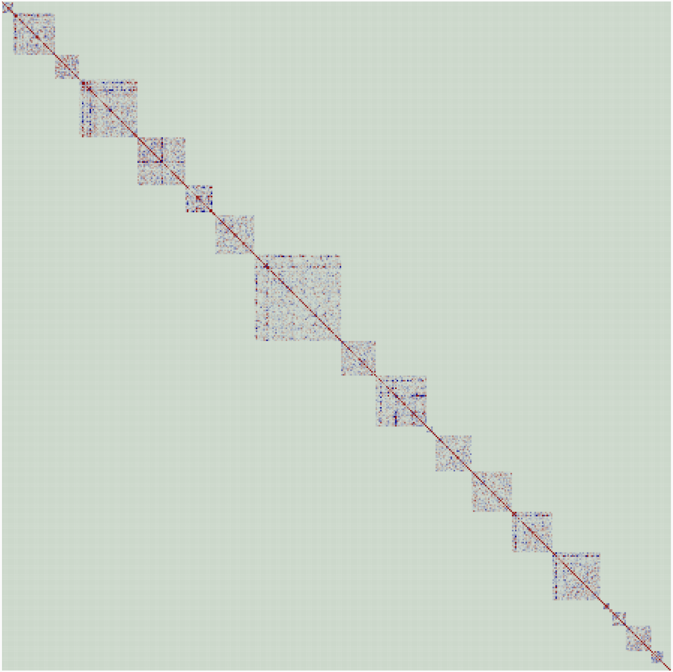


⟵ diagonalized psd (⊂ 448 × 448)
Original psd constraint (4 641 × 4 641)

The main result

Theorem (Kaluba-Kielak-Nowak, 2018)

Let $G_n = \mathrm{SL}_n(\mathbb{Z})$ *(or* $G_n = \mathrm{Aut}(F_n)$*). There exists* $\lambda_n > 0$ *such that*

$$\Delta_n^2 - \lambda_n \Delta_n \in \Sigma^2 I[G_n],$$

i.e. G *has property (T) for for all* $n \geqslant 3$ *(*$n \geqslant 6$*, respectively), with*

$$\sqrt{\frac{2\lambda}{|S|}} \leqslant \kappa(G, S).$$

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- Hope that the remainder is a sum of squares. □