Recall: Every intransitive group is a sub-direct product of its transitive parts.

Is there a universal description for imprimitive groups?

Defn: Let G, H be groups. Let $\psi : H \to \mathrm{Aut}(G)$ be a homomorphism. Then

$$G \rtimes_\psi H = \langle (g,h) \in G \times H, \cdot_\psi, (1,1) \rangle$$

is a group with

$$(g,h) \cdot_\psi (a,b) = (g \cdot \psi(h)(a), hb)$$

$(1,1) = (g,h) \cdot (a,b) \Rightarrow b = h^{-1}, \quad a = \psi(h^{-1})(g^{-1})$

$(g,h)(\psi(h^{-1})(g^{-1}), h^{-1}) = (g \cdot \psi(hh^{-1})(g^{-1}), hh^{-1}) = (1,1)$

Note $G \triangleleft G \rtimes_\psi H$.

Defn: Wreath product of G and $H < \mathrm{Sym}(n)$

permutation group

$$G \wr H := G^n \rtimes H$$

the natural action of H on coordinates of G^n.

$$((g_1, \dots, g_n), h) \cdot ((a_1, \dots, a_n), b) =$$

$$= ((g_1, \dots, g_n) \cdot (a_1, \dots, a_n)^{h^{-1}}, h \cdot b) =$$

$$= ((g_1 \cdot a_{h(1)}, g_2 \cdot a_{h(2)}, \dots, g_n \cdot a_{h(n)}), h \cdot b).$$

Suppose that $G \curvearrowright \Omega \Rightarrow G \wr H \curvearrowright \bigsqcup_n \Omega$

$H < Sym(n)$

i-th copy acts on i-th copy of Ω

H permutes Gs $\rightarrow$ H permutes copies of Ω.

$\boxed{\text{the imprimitive action of } G \wr H}$

Block system? $\{\Omega \times \{1\}, \Omega \times \{2\}, \dots\}$

Theorem: Let G be a transitive, imprimitive group. Let $\mathcal{B}$ be a non-trivial block system for G.

Let $1 \in B \in \mathcal{B}$ and let $T = Stab_G(B) < G$.

Let $\psi : G \longrightarrow Sym(\mathcal{B})$ (action homomorphism)

$\varphi : T \longrightarrow Sym(B)$ (action homomorphism)

Then $G \xhookrightarrow{\mu} \varphi(T) \wr \psi(G)$

The monomorphism is given as follows:

let r_i be the coset reps for $T\backslash G$.

then G permutes the cosets as

$$(r_i, g) \longmapsto \overline{r_i \cdot g} = r_j \text{ for some } j.$$
$$(j = i^{\psi(g)})$$

$\tilde{g}_i = r_i \cdot g \cdot \overline{r_i \cdot g}^{-1}$ stabilizes $B_1 \Rightarrow g_i \in T$.

$$\mu(g) = ((\varphi(\tilde{g}_1), \dots, \varphi(\tilde{g}_n)), \psi(g))$$

(that's almost correct)

$$\mu(g) = ((\varphi(\tilde{g}_{1^{\psi(g)^{-1}}}), \dots, \varphi(\tilde{g}_{n^{\psi(g)^{-1}}})), \psi(g)).$$

__Proof:__ Mono - if $\psi(g) = 1 \Rightarrow g$ stabilizes $\mathcal{B} \Rightarrow g \in T$

if every $\varphi(\tilde{g}_i) = 1 \Rightarrow g$ acts trivially on every B_i

$\Rightarrow g$ acts trivially on Ω.

that μ is a homomorphism

$\Rightarrow$ exercise.

Corollary: Every imprimitive group G is a __subgroup__ of the wreath product of perm. groups of smaller degree.

Classification of primitive groups

Lemma:

If $G \curvearrowright \Omega$ transitively and $N \trianglelefteq G$ then the orbits of $N \curvearrowright \Omega$ form a block system.

Proof: Δ - N-orbit, $g \in G$.

let $\Delta = x^N$, $\delta, \gamma \in \Delta \Rightarrow \delta^n = \gamma$

We want to show that δ^g and γ^g differ by $m \in N$ for every $g \in G$.

$$\gamma^g = \delta^{ng} = (\delta^g)^{\underbrace{g^{-1}ng}} = (\delta^g)^m \quad m \in N.$$

If Δ^g were not the whole N-orbit of x^g then $(\Delta^g)^{g^{-1}} = \Delta$ wouldn't be the whole orbit of x.

$\lightning$

$\square$.

Corollary: If G is primitive then N acts transitively.

Defn:

$\{1\} \neq N \trianglelefteq G$ is called <u>minimally normal</u> if the only normal in G proper subgroup of N is $\{1\}$.

$$(M \trianglelefteq G \ \& \ M \lneq N \Rightarrow M = \{1\})$$

Lemma:

Minimally normal subgroups are of the form

$$\mathcal{N} = \bigoplus_k T$$

where T is a simple group.

Proof:

Let $M \triangleleft \mathcal{N}$ be the first proper subgroup in the composition series of $\mathcal{N}$.

$\Rightarrow {}_M\backslash \mathcal{N} =: T$ is simple.

Consider M^G - the orbit of M under G. (every $M^g = g^{-1} M g < \mathcal{N}$)

Let $\mathcal{D} = \prod_{S_i \in M^G} {}_{S_i}\backslash \mathcal{N}$ $\quad S_i = g_i^{-1} M g_i$ — fixed set of reps

and let $\varphi: \mathcal{N} \curvearrowright \mathcal{D}$ (action homomorphism).

$$n \longmapsto (S_1 n, S_2 n, \dots, S_k n).$$

$\varphi(n) = 1 \iff n \in S_1$ & $n \in S_2$ & ... & $n \in S_k$

$$\operatorname{Ker} \varphi = \bigcap_i g_i^{-1} M g_i \triangleleft G$$

$\Rightarrow \ker \varphi = 1$ by minimality of $\mathcal{N}$

$\Rightarrow \varphi$ - injective and

$\mathcal{N} \curvearrowright$ via φ intransitively on $\mathcal{D}$ (each ${}_{S_i}\backslash \mathcal{N}$ is a separate orbit).

But ${}_{S_i}\backslash \mathcal{N} \cong T$ is simple $\Rightarrow$ the amalgamating group is trivial $\Rightarrow$

$\mathcal{N} \cong$ an honest product.

Definition: The socle of G is the subgroup generated by minimally normal subgroups:

$$\operatorname{soc}(G) = \langle N \mid N \trianglelefteq G \text{ - minimally} \rangle.$$

Lemma:

$$\operatorname{soc}(G) = \bigoplus_i N_i \qquad N_i \text{ - minimally normal}$$

Proof:

If $H = \langle N, M \rangle$ (subgroups) and $N \cap M = \langle 1 \rangle \Rightarrow H \cong N \rtimes M$

If $N \trianglelefteq H$ & $M \trianglelefteq H \Rightarrow H \cong N \oplus M$.

Let M - maximal subgroup of $\operatorname{soc}(G)$ which is a product of minimally normal subgroups.

If $M \neq \operatorname{soc}(G) \Rightarrow \exists\, N \trianglelefteq G$ s.t. $N \not\leq M$.
minimally normal.

then $N \cap M \trianglelefteq G$

by minimality of N: $N \cap M = \langle 1 \rangle$

$\Rightarrow \langle N, M \rangle = N \times M$

$\lightning$. $\square$

Lemma:

Let $G \curvearrowright \Omega$ primitively. Then either

$(*)$ $\operatorname{soc}(G) \cong T^m$ is minimally normal or

$(**)$ $\operatorname{soc}(G) = N \times M$ where $N \cong M$ is
- minimally normal and
- non-abelian

(in either case we have $\operatorname{soc}(G) \cong T^m$ and we say that it is homogeneous of type T.

Types of Socles: $G \curvearrowright \Omega$ primitively

- $Soc(G) \cong \bigoplus_m T$ ← homogeneous of type T

  ↳ T - abelian i.e. $T \cong C_p$ - cyclic of order p.

  $\Rightarrow$ primitive $G \cong Soc(G) \rtimes Stab_G(1)$

Proof:

$Soc(G)$ - abelian, minimally normal $\Rightarrow$

$Soc(G) \cong C_p^m \cong \mathbb{F}_p^m$. By transitivity & faithfullness

p - prime

$|\Omega| = p^m$.

Let $S = Stab_G(1)$ ← by primitiveness S is a maximal subgroup, but

(in primitive actions normal subgrps act transitively!)

$Soc(G) \not\leq S$ ($Soc(G) \curvearrowright \Omega$ transitively!)

$\Rightarrow G = Soc(G)S$.

Since $Soc(G)$ is abelian $\Rightarrow \underbrace{S \cap Soc(G) \triangleleft Soc(G)}$.

$Soc(G) \triangleleft G \Rightarrow S \cap Soc(G) \triangleleft S$. $S \cap Soc(G) \triangleleft G \Rightarrow S \cap Soc(G) = \langle 1 \rangle$.

$\Rightarrow G \cong Soc(G) \rtimes S$.

$\square$

Note: the action of G on Ω is through an affine map where each element of S acts through a matrix representation

$$S \longrightarrow GL(m, \mathbb{F}_p)$$

and $Soc(G)$ corresponds to translations.

- $Soc(G)$ is non-abelian.

↳ If $Z(soc(G)) = \langle 1 \rangle \Rightarrow G \leq Aut(soc(G))$.

<u>Proof:</u> $G \curvearrowright$ on $Soc(G)$ by conjugation;

$C_G(soc(G))$ — the kernel of the action.

Let $H < C_G(soc(G))$ be a minimally normal subgroup of G

$\Rightarrow H < soc(G)$ and therefore $H < Z(soc(G)) = \langle 1 \rangle$.

$\Rightarrow G \hookrightarrow Aut(soc(G))$.

<u>Lemma:</u> for T - simple $Aut(T^m) = Aut(T) \wr Sym(m)$

$\square$

<u>Corollary:</u> If $Z(soc(G)) = 1$ & $soc(G)$ - non abelian

$\Rightarrow G \hookrightarrow Aut(T) \wr Sym(m)$.

$(1,h)\cdot(g,1) = (g,1)$
$(g,1)\cdot(1,h) = (g^h, h)$.

Product action of $G \wr H$:

$$G \leq Sym(\Omega), \quad H \leq Sym(\Delta) \Rightarrow G \wr H \curvearrowright \Omega^{\Delta} \cong \Omega^{|\Delta|}$$

$|\Delta|$-tuples of elts from Ω

$d = |\Delta|$ $\quad G^d \curvearrowright \Omega^d$ "dimension-wise"

$H \curvearrowright \Omega^d$ "permuting the dimensions"

$$(\omega_1, \dots, \omega_{|\Delta|})^{(g,h)} = \left(\omega_{1^{h^{-1}}}^{(g_1^h)}, \dots, \omega_{|\Delta|^{h^{-1}}}^{(g_{|\Delta|}^h)}\right)$$

$\boxed{\text{the product action of } G \wr H}$.

Let $\mathcal{D} < T^m$ be the image of

$$T \hookrightarrow T^m \quad g \mapsto (g, \dots, g). \quad \text{(diagonal embedding)}.$$

$$T^m \curvearrowright \underbrace{\mathcal{D} \backslash T^m}_{\Omega}$$ (action homomorphism into $\mathrm{Sym}(|\mathcal{D}\backslash T^m|)$ of degree $n = |T|^{m-1}$.)

$\mathcal{N} = N_{\mathrm{Sym}(n)}(T^m) \curvearrowright T^m$ by conjugation

$\mathcal{N} \triangleleft \mathrm{Aut}(T^m)$

However we don't necessarily have $G < \mathcal{N}$.
In the case this happens we say that $G \curvearrowright \Omega$ in <u>diagonal type.</u>

<u>Lemma:</u>
G of diagonal type is primitive iff $m = 2$ or $G \curvearrowright T^m$ is primitive.

Theorem: (Scott - O'Nan theorem).

$G \curvearrowright \Omega$ primitively, faithfully with $|\Omega| = n$.
Let $H = soc(G)$ and assume that $H = T^m$ is of type T. Then one of the points below describes the action:

1) T is abelian of order p, $n = p^m$,
$G \cong H \rtimes Stab_G(x)$ $(x \in \Omega)$
$G \curvearrowright \Omega$ through an affine action.

2) $m = 1$, $H \triangleleft G \leq Aut(H)$ "G is almost simple".

3) $m \geq 2$, $n = |T|^{m-1}$, $G \leq Aut(T) \wr Sym(m)$ and either

3a) $m = 2$ $G \curvearrowright \{T_1, T_2\}$ intransitively

3b) $m \geq 2$ $G \curvearrowright \{T_1, \dots, T_m\}$ primitively

the action of G on Ω is of the diagonal type.

4) $m = rs$, $s > 1$ then

$G \leq A \wr B$ where $A \wr B$ acts through the product action. ($\Omega = \prod_s \Delta$, $|\Omega| = |\Delta|^s$)
$A \curvearrowright$ primitively on $|\Delta|$ pts
$B \curvearrowright$ transitively on s pts.

4a) A is of type 3a with $soc(A) = T^2$ (i.e. $r = 2$)
4b) of type 3b with $soc(A) = T^r$
4c) of type 2 (i.e. $r = 1$, $s = m$).

5) "Twisted wreath type". H is freely and $n = |T^m|$.
$\mathrm{Stab}_G(x)$ is a transitive subgroup of $\mathrm{Sym}(m)$.
(note: this type occurs only for groups of order 60^6).