Imprimitive groups:

- $G \curvearrowright \Omega$ transitively

Defn:

Let $\mathcal{B} = \{B_1, \dots, B_n\}$ $B_i \subset \Omega$; $B_i \cap B_j = \emptyset$ for $i \neq j$
$\bigcup B_i = \Omega$ (i.e. $\mathcal{B}$ is a partition of Ω).

$\mathcal{B}$ is a block system for $G \curvearrowright \Omega$ when
$B_i^g \in \mathcal{B}$ $\forall g \in G$
(i.e. $\mathcal{B}$ is G-invariant).

Ex: trivial block systems: $\mathcal{B}_1 = \{\{i\} : i \in \Omega\}$

$\mathcal{B}_\infty = \{\Omega\}$.

Ex: $G = \langle (1,2,3,4) \rangle$
$\mathcal{B} = \{\{1,3\}, \{2,4\}\}$

Defn: We say that $G \curvearrowright \Omega$ imprimitively iff $G \curvearrowright$ transitively and admits a non-trivial block system.

(Otherwise we say that $G \curvearrowright$ primitively).

Lemma: Let $\mathcal{B} = \{B_1, \dots, B_k\}$ be a block system for $G \curvearrowright \Omega$. The action of G on blocks is transitive.

Proof: transitive $=$ $\forall 1 \leq i,j \leq k$ $\exists g \in G$ s.t. $B_i^g = B_j$.

Let $\delta \in B_i$ and $\gamma \in B_j$. since $G \curvearrowright \Omega$ transitively

$\Rightarrow \exists g \in G$ s.t. $\delta^g = \gamma$. the same g moves B_i to B_j.

Corollary:

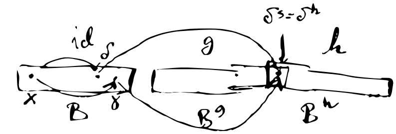
- $|\Omega| = |\mathcal{B}| \cdot |B_1|$
- Block system is determined by a single block.
- If $|G| = p$ (prime); $G \curvearrowright \Omega$ transitively, then $G \curvearrowright$ primitively.

Lemma: Suppose $G \curvearrowright \Omega$ transitively and let $S = \mathrm{Stab}_G(x)$ for some $x \in \Omega$.
then there is a bijection between subgroups
$\{T : S \leq T \leq G\}$ and
block systems $\mathcal{B} = \{B_1, \dots, B_k\}$ for $G \curvearrowright \Omega$.

The bijection is given by
$$x, T \rightsquigarrow x^T =: B_1 \rightsquigarrow \mathcal{B} = B_1^G.$$

The inverse is simply
$$\mathcal{B} = \{B_1, \dots, B_k\} \rightsquigarrow \mathrm{Stab}_G(B_1).$$

Proof:

Let $S \le T \le G$ and pick $x \in \Omega$ 

Set $B = x^T$, $\mathcal{B} = B^G$.

Claim: $\mathcal{B}$ is a block system for $G \curvearrowright \Omega$.

Let $B^g \cap B^h \neq \emptyset$ i.e. $\delta^g = \gamma^h$ for some $\delta, \gamma \in B$.

Since $B = x^T \Rightarrow \delta = x^a$, $\gamma = x^b$ for some $a, b \in T$.

$\Rightarrow x^{a \cdot g} = x^{bh}$ i.e $x^{a \cdot g \cdot h^{-1} \cdot b^{-1}} = x$

$\Rightarrow a \cdot g \cdot h^{-1} \cdot b^{-1} \in Stab_G(x) \le T$

$\Rightarrow gh^{-1} \in T$.

Since T stabilizes B $B^{gh^{-1}} \cap B = B$ i.e. $B^g = B^h$

G-invariance is obvious by the definition of $\mathcal{B}$

Let $\mathcal{B}$ be a block system, $x \in B_1 \in \mathcal{B}$.

If g fixes $x_1 \Rightarrow x_1 \in B_1 \cap B_1^g \neq \emptyset \Rightarrow B_1^g = B_1$

$$Stab_G(x) \le Stab_G(B_1).$$

Let $\delta \in B_1 \Rightarrow x^g = \delta$ (by transitivity) $\Rightarrow \delta \in B_1^g \cap B_1 \neq \emptyset$

$\Rightarrow g \in Stab_G(x)$. Therefore $x^{Stab_G(B_1)} = B_1$.

To make that the maps are inverse of each other we need to check:

$$Stab_G\left(x^{Stab_G(B_1)}\right) \overset{?}{=} Stab_G(B_1) =: T.$$

If $g \in G$ s.t. $(x^T)^g \Rightarrow \forall t \in T \; \exists t' \in T$ s.t. $x^{tg} = x^{t'}$

$\Rightarrow x^{tgt'^{-1}} = x \Rightarrow tgt'^{-1} \in T \Rightarrow g \in T$.

□

Definition:

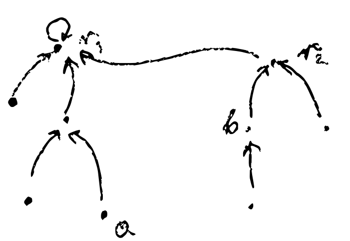
A subgroup $S < G$ is called maximal if $S \neq G$ and there is no subgroup $T < G$ s.t. $S \lneq T$.

Corollary:

- A transitive group G is primitive iff a point stabilizer is a maximal subgroup.
- Subgroup $S < G$ is maximal iff $G \curvearrowright S \backslash G$ is primitive.

Finding blocks:

- Let $X \subset \Omega$ be a subset (a seed) we want to be $\subset B_1$.
- Start with $B_X = X$, $\{B_y = \{y\}$ for $y \in \Omega \setminus X\}$
- Act via $G = \langle S \rangle$ on each of B_i and merge those which intersect.
- Each B_i is represented by a unique element $\rightarrow$ the representative
- for each $x \in \Omega$ we store the representative of B_i which x belongs to.

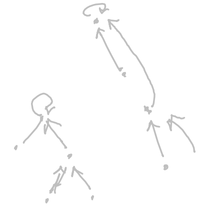


$v_1 < v_2$

ALGORITHM: Union!

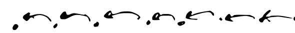
Input: C_1 – a subset of Ω
C_2 – a subset of Ω

Output: the union of C_1 and C_2



```
begin
    r_1 = representative(C_1)
    r_2 = representative(C_2)
    if r_1 ≠ r_2
        set_representative!(r_2, r_1)
    end
    return C_1
end
```

path halving:
```
while parent(x) ≠ x
    x.parent = parent(parent(x))
    x = parent(x)
end
```



ALGORITHM: Block_system

INPUT:
- S – a generating set for $G = \langle S \rangle$
- Ω – a set with G-action
- B – a subset of Ω

OUTPUT: $\mathcal{B}$ – the finest Block system for $G \curvearrowright \Omega$ s.t. $B \subset B_1$

```
begin
    for x ∈ Ω
        set_representative!(x, x)
    end
    queue = [] # a queue of points that have changed
    for x ∈ B
        set_representative!(x, first(B))
        push!(queue, x)
    end
```

initialize the initial partition

```
    while !isempty(queue)
        x = pop!(queue)
        y = representative(x)
        for s ∈ S
            α = representative(x^s)
            β = representative(y^s)
            if α ≠ β
                Union!(α, β)
                push β to queue
            end
        end
    end
    return r
end
```

Theorem:

The algorithm converges to a block system for $G \curvearrowright \Omega$

Proof: Since we're taking only unions the returned partition contains B in one of its blocks.

Since every Union! moves up in the lattice of all partitions of Ω and $\{\Omega\}$ - the trivial block system is the maximal element the algorithm has to stop.

We need to prove that the refined partition $\mathcal{B}$ is $\mathcal{G}$-invariant i.e. $\forall B \in \mathcal{B}\ \forall x, y \in B\ \forall g \in \mathcal{G}\ x^g, y^g \in B' \in \mathcal{B}$.

Observe: • It's enough to check this for $y = r(x)$

• It's enough to check this for $g \in S$.

$$\forall B \in \mathcal{B}\ \forall x \in B\ \forall s \in S \quad r(x^s) = r(r(x)^s).$$

$$\forall x \in \Omega\ \forall s \in S \quad r(x^s) = r(r(x)^s). \quad (*)$$

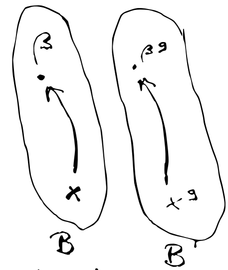
• It's enough to enforce the condition only for pts which changed label → it's the queue.

Had we added all points whose representative is β we would be ok. But only β was added...

Let $x \in \Omega$: representative$(x) = \beta \neq x$

$\Rightarrow x$ was on the queue

$\Rightarrow (*)$ is satisfied for (x, β).



If it happens that we change $r(\beta) = \alpha$ then $(*)$ is enforced for (β, α), so $r(x^g) = r(\beta^g) =$

$= r(r(\beta)^g) = r(\alpha^g)$ $\square$.

$=$

Lemma: Let $1 \in B_1 \in \mathcal{B}$ ← block system for $\mathcal{G} \curvearrowright \Omega$.

B_1 is union of orbits of $\mathrm{Stab}_{\mathcal{G}}(1)$.

Proof: If $g \in \mathrm{Stab}_{\mathcal{G}}(1)$, $1 \neq \alpha \in B_1 \Rightarrow$

$\{1, \alpha\}^g = \{1, \alpha^g\} \subset B_1^g = B_1 \Rightarrow \alpha^{\mathrm{Stab}_{\mathcal{G}}(1)} \in B_1$.

Corollary: Any block that contains $\{1, \alpha\}$ contains also $\{1\} \cup \alpha^{\mathrm{Stab}_G(1)}$.

Practical tip: We don't need to know $\mathrm{Stab}_G(1)$ exactly!

It's enough to find a few random elements from $\mathrm{Stab}_G(1)$ to compute the orbit of α.

(e.g. Schreier generators from the transversal)

Recall: Every intransitive group is a sub-direct product of its transitive parts.

Is there a universal description for imprimitive groups?

Defn: Let G, H be groups. Let $\psi: H \to \mathrm{Aut}(G)$ be a homomorphism. Then

$$G \rtimes_\psi H = \langle (g,h) \in G \times H, \cdot_\psi, (1,1) \rangle$$

is a group with

$$(g,h) \cdot_\psi (a,b) = (g \cdot \psi(b)(a), hb)$$

$$(1,1) = (g,h) \cdot (a,b) \Rightarrow b = h^{-1}, \quad a = \psi(h)(g^{-1})$$

Note: $G \lhd G \rtimes_\psi H$.