

Given $G < \text{Sym}(U)$ split it into

$N \triangleleft G$ and G/N i.e.

$$1 \longrightarrow N \xrightarrow{\quad} G \xrightarrow{\varphi} G/N \longrightarrow 1$$

\downarrow
ker φ

faithfully \swarrow disjoint union of G -invariant sets

$$G \curvearrowright \Omega = \Delta \sqcup \Gamma$$

$$\Rightarrow G \xrightarrow{\alpha} \text{Sym}(\Delta)$$

action homomorphisms

$$G \xrightarrow{\beta} \text{Sym}(\Gamma)$$

satisfy $\ker \alpha \cap \ker \beta = \langle 1 \rangle$.

$$\text{im } \alpha = A < \text{Sym}(\Delta)$$

$$\text{im } \beta = B < \text{Sym}(\Gamma)$$

$$\varphi: G \longrightarrow A \times B$$

$$g \longmapsto (\alpha(g), \beta(g))$$

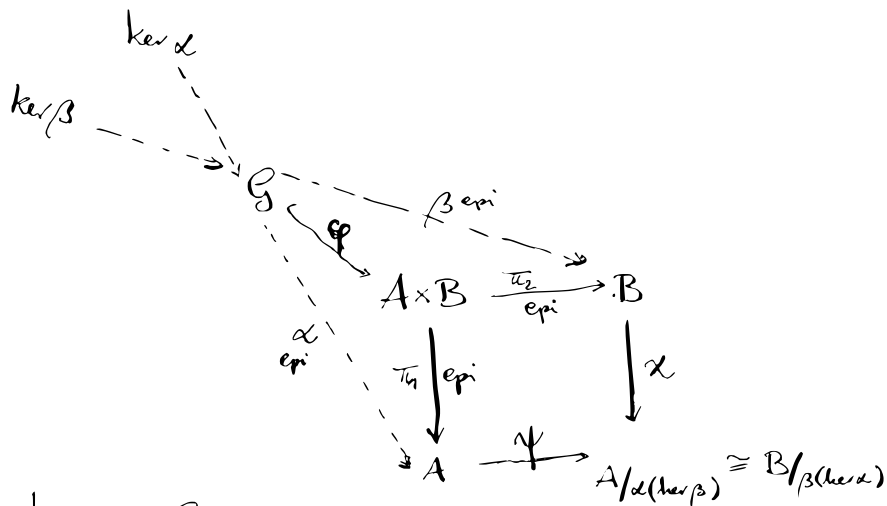
$$G \xrightarrow{\varphi} A \times B \quad \ker \varphi = \langle 1 \rangle.$$

$$A \times B \xrightarrow[\text{epi}]{\pi_2} B$$

$$\text{epi} \downarrow \pi_1$$

$$A$$

Defn. We say that G is (isomorphic to) a sub-direct product.



$$\ker \alpha \triangleleft G \Rightarrow \beta(\ker \alpha) \triangleleft \beta(G) = B$$

$$\alpha(\ker \beta) \triangleleft \alpha(G) = A$$

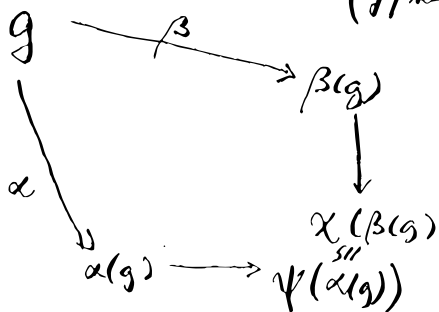
- Aim: describe $\text{im } \varphi$ in terms of A and B .

$$\alpha(G)/\alpha(\ker \beta) = (G/\ker \alpha)/\alpha(\ker \beta) \stackrel{\downarrow}{=}$$

$$= G/\langle \ker \alpha, \ker \beta \rangle =$$

$$= G/\langle \ker \beta, \ker \alpha \rangle =$$

$$= (G/\ker \beta)/\beta(\ker \alpha) = \beta(G)/\beta(\ker \alpha)$$



← explicit isomorphism above

$$\zeta: A/\alpha(\ker \beta) \rightarrow B/\beta(\ker \alpha)$$

$$\zeta(\psi(\alpha(g))) = \alpha(\beta(g)).$$

thus we can characterize

$$\varphi(g) \in A \times B \text{ as}$$

$$\{(a, b) \in A \times B : \zeta(\varphi(a)) = \chi(b)\}$$

$\times E$

Definition: Pullback (direct product with amalgamation, external subdirect product)

A, B two groups s.t.

$$E \triangleleft A, E \triangleleft B \text{ and } \zeta: A/E \xrightarrow{\cong} B/E.$$

$$\text{then } A \otimes_{\zeta} B = \{(a, b) \in A \times B : \zeta(aE) = bE\}$$

From mathematical perspective it's easier to look at

$$\begin{array}{ccc} A \times_{\alpha} B & \longrightarrow & B \\ \downarrow & & \downarrow \alpha \\ A & \xrightarrow{\psi} & Q \end{array}$$

and consider $A \times_{\alpha} B = \{(a, b) \in A \times B : \psi(a) = \alpha(b)\}$

however this hides from us ζ

which is very visible when working with permutation groups explicitly.

Let $\psi: A \rightarrow \mathbb{Q}$, $\chi: B \rightarrow \mathbb{Q}$ be epimorphisms.

And let

$$A \times_{\mathbb{Q}} B \xrightarrow{\pi_2} B$$

$$A \times_{\mathbb{Q}} B = \{(a, b) \in A \times B : \psi(a) = \chi(b)\}$$

$$\downarrow \pi_1 \quad \downarrow \chi$$

be their sub-direct product

$$A \xrightarrow{\psi} \mathbb{Q}$$

Suppose that there exist G s.t.

$$\begin{array}{ccc} G & \xrightarrow{\beta} & B \\ \alpha \downarrow & & \downarrow \chi \\ A & \xrightarrow{\psi} & \mathbb{Q} \end{array} \quad \text{commutes, i.e. } \forall g \in G \quad \psi(\alpha(g)) = \chi(\beta(g))$$

Lemma:

Then there exists a unique homomorphism

$$u: G \rightarrow A \times_{\mathbb{Q}} B \quad \text{s.t.}$$

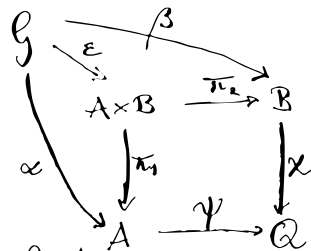
$$\begin{array}{ccccc} G & & & & \\ & \searrow \beta & & & \\ & & A \times_{\mathbb{Q}} B & \xrightarrow{\pi_2} & B \\ & \searrow u & & & \downarrow \chi \\ & & \downarrow \pi_1 & & \\ & & A & \xrightarrow{\psi} & \mathbb{Q} \\ & \searrow \alpha & & & \\ & & & & \end{array}$$

commutes.

Proof: There are two statements here:

1) $\exists \mu: G \rightarrow A \times_{\alpha} B :$

Let $\varepsilon: G \rightarrow A \times B$
 $g \mapsto (\alpha(g), \beta(g))$



Observe:

$$\pi_1(\varepsilon(g)) = \alpha(g) \quad \& \quad \pi_2(\varepsilon(g)) = \beta(g)$$

$$\text{hence } \psi(\pi_1(\varepsilon(g))) = \psi(\alpha(g)) = \chi(\beta(g)) = \chi(\pi_2(\varepsilon(g)))$$

$\Rightarrow \varepsilon(g)$ satisfies the pullback condition in $A \times B$

$$\Rightarrow \varepsilon(G) \leq A \times_{\alpha} B$$

$\mu: G \rightarrow A \times_{\alpha} B$ is given by ε .
 (co-restriction)

2) μ is unique

suppose that $\mu': G \rightarrow A \times_{\alpha} B$ is another such map.

i.e. $\mu'(g) = (a', g')$ and the diagram above is commutative.

By looking at the triangles we see that

$$a' = \pi_1(\mu'(g)) = \alpha(g)$$

$$b' = \pi_2(\mu'(g)) = \beta(g)$$

$$\text{ie. } \mu' = \mu.$$

Corollary:

Every intransitive perm. group is a sub-direct product of two perm. groups of smaller degree.

Example: $A = \text{Sym}(3) = B$

$$\begin{array}{ccc} & B & \\ & \downarrow \chi & \\ A & \xrightarrow{\psi} & Q \end{array} \quad \text{What are options for } Q?$$

1) $Q = \{1\}$

$$\begin{aligned} \Rightarrow A \times_Q B &= \{ (a, b) \in A \times B \text{ s.t. } \psi(a) = \text{id} = \chi(b) \} \\ &= A \times B \cong \langle (1, 2), (1, 2, 3), (4, 5), (4, 5, 6) \rangle \end{aligned}$$

2) $Q = \text{Sym}(3)$ i.e. ψ and χ are isomorphisms

$$\begin{array}{l} \text{id.} \\ \chi((4, 5)) = (1, 2) \\ \chi((4, 5, 6)) = (1, 2, 3) \end{array}$$

$$\begin{aligned} \Rightarrow A \times_Q B &= \{ (a, b) \in A \times B \text{ s.t. } a = \chi(b) \} \\ &= \langle (1, 2)(4, 5), (1, 2, 3)(4, 5, 6) \rangle \cong \text{Sym}(3) \end{aligned}$$

3) $Q = C_2 = \langle (4, 5) \rangle$ i.e. $\psi = \text{id.}$, $\chi((4, 5)) = (1, 2)$
 $\chi((4, 5, 6)) = ()$.

$$\begin{aligned} \Rightarrow A \times_Q B &= \{ (a, b) \in A \times B \text{ s.t. } a = \chi(b) \} = \\ &= \langle (1, 2)(4, 5), (1, 2, 3), (4, 5, 6) \rangle \end{aligned}$$