Given $G < \mathrm{Sym}(n)$ split it into

$N \triangleleft G$ and G/N i.e.

$$1 \longrightarrow \underset{\underset{\ker\varphi}{\shortparallel}}{N} \longrightarrow G \xrightarrow{\varphi} G/N \longrightarrow 1$$

faithfully

$$G \curvearrowright \Omega = \Delta \sqcup \Gamma$$

disjoint union of G-invariant sets

$$\Rightarrow \quad G \xrightarrow{\alpha} \mathrm{Sym}(\Delta)$$
$$G \xrightarrow{\beta} \mathrm{Sym}(\Gamma)$$

action homomorphisms

satisfy $\ker\alpha \cap \ker\beta = \langle 1 \rangle$.

$$\mathrm{im}\,\alpha = A < \mathrm{Sym}(\Delta)$$
$$\mathrm{im}\,\beta = B < \mathrm{Sym}(\Gamma)$$

$$\varphi : G \longrightarrow A \times B$$
$$g \longmapsto (\alpha(g), \beta(g))$$

$\ker\varphi = \langle 1 \rangle$.

$$\begin{tikzcd} G \arrow[d, hook, "\varphi", "\text{inj.}"'] & \\ A \times B \arrow[r, "\pi_2", "\text{epi}"'] \arrow[d, "\pi_1", "\text{epi}"'] & B \\ A & \end{tikzcd}$$

<u>Defn.</u> We say that G is (isomorphic to) a <u>sub-direct product</u>.

$$\begin{tikzcd}
\ker\beta \arrow[dr, dashed] & \ker\alpha \arrow[d, dashed] & & \\
& G \arrow[r, "\varphi"] \arrow[rrr, dashed, bend left, "\beta \text{ epi}"] \arrow[drr, dashed, "\alpha \text{ epi}"'] & A\times B \arrow[rr, "\pi_2", "\text{epi}"'] \arrow[d, "\pi_1", "\text{epi}"'] & & B \arrow[d, "\chi"] \\
& & A \arrow[rr, "\psi"] & & A/\alpha(\ker\beta) \cong B/\beta(\ker\alpha)
\end{tikzcd}$$

$$\ker\alpha \triangleleft G \Rightarrow \beta(\ker\alpha) \triangleleft \beta(G) = B$$

$$\alpha(\ker\beta) \triangleleft \alpha(G) = A$$

- Aim: describe $\operatorname{im}\varphi$ in terms of A and B.

$$\alpha(G)/\alpha(\ker\beta) = (G/\ker\alpha)/\alpha(\ker\beta) \overset{\checkmark}{\cong}$$

$$= G/\langle \ker\alpha, \ker\beta\rangle =$$

$$= G/\langle \ker\beta, \ker\alpha\rangle =$$

$$= (G/\ker\beta)/\beta(\ker\alpha) = \beta(G)/\beta(\ker\alpha)$$

$$\begin{tikzcd}
g \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \beta(g) \arrow[d] \\
\alpha(g) \arrow[r] & \chi(\beta(g)) \cong \psi(\alpha(g))
\end{tikzcd}$$

$\leftarrow$ explicit isomorphism above

$$\zeta : A/\alpha(\ker\beta) \to B/\beta(\ker\alpha)$$

$$\zeta(\psi(\alpha(g))) = \chi(\beta(g)).$$

thus we can characterize

$\varphi(G) \subset A \times B$ as

$$\{ (a,b) \in A \times B : \zeta(\psi(a)) = \chi(b) \}$$

<u>Definition</u> : <u>Pullback</u> (direct product with amalgamation, external subdirect product)

A, B two groups s.t.

$D \triangleleft A$, $E \triangleleft B$ and $\zeta : A/D \xrightarrow{\cong} B/E$.

then $A \oplus_\zeta B = \{ (a,b) \in A \times B : \zeta(aD) = bE \}$

From mathematical perspective it's easier to look at

$$\begin{tikzcd} A \times_Q B \arrow[r] \arrow[d] & B \arrow[d, "\chi"] \\ A \arrow[r, "\psi"] & Q \end{tikzcd}$$

and consider $A \times_Q B = \{ (a,b) \in A \times B : \psi(a) = \chi(b) \}$

however this hides from us ζ which is very visible when working with permutation groups explicitly.

Let $\psi: A \to Q$, $\chi: B \to Q$ be epimorphisms.

And let

$A \times_Q B = \{ (a,b) \in A \times B : \psi(a) = \chi(b) \}$

be their sub-direct product

$$\begin{tikzcd} A \times_Q B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "\chi"] \\ A \arrow[r, "\psi"'] & Q \end{tikzcd}$$

Suppose that there exist G s.t.

$$\begin{tikzcd} G \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[d, "\chi"] \\ A \arrow[r, "\psi"'] & Q \end{tikzcd}$$

commutes, i.e. $\forall g \in G$

$$\psi(\alpha(g)) = \chi(\beta(g))$$

<u>Lemma:</u>

Then there exists a unique homomorphism

$\mu: G \to A \times_Q B$ s.t.

$$\begin{tikzcd} G \arrow[dr, "\mu"] \arrow[drr, "\beta"] \arrow[ddr, "\alpha"'] & & \\ & A \times_Q B \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & B \arrow[d, "\chi"] \\ & A \arrow[r, "\psi"'] & Q \end{tikzcd}$$

commutes.

Proof: There are two statements here:

1) $\exists\, \mu : G \to A \times_Q B$:

$$\begin{tikzcd}
G \arrow[dr, "\varepsilon"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "\chi"] \\
& A \arrow[r, "\psi"] & Q
\end{tikzcd}$$

Let $\varepsilon : G \to A \times B$
$\qquad g \longmapsto (\alpha(g), \beta(g))$

Observe:

$$\pi_1(\varepsilon(g)) = \alpha(g) \;\&\; \pi_2(\varepsilon(g)) = \beta(g)$$

hence $\quad \psi(\pi_1(\varepsilon(g))) = \psi(\alpha(g)) = \chi(\beta(g)) =$
$$= \chi(\pi_2(\varepsilon(g)))$$

$\Rightarrow \varepsilon(g)$ satisfies the pullback condition in $A \times B$

$\Rightarrow \varepsilon(G) \leq A \times_Q B$

$\mu : G \to A \times_Q B$ is given by ε.
(co-restriction)

2) μ is unique

Suppose that $\mu' : G \to A \times_Q B$ is another such map.

i.e. $\mu'(g) = (a', g')$ and the diagram above is commutative.

By looking at the triangles we see that

$$a' = \pi_1(\mu'(g)) = \alpha(g)$$
$$b' = \pi_2(\mu'(g)) = \beta(g) \qquad \text{ie. } \mu' = \mu.$$

Corollary:

Every intransitive perm. group is a sub-direct product of two perm. groups of smaller degree.

Example: $A = \mathrm{Sym}(3) = B$

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle \chi} \\ A & \xrightarrow{\psi} & Q \end{array}$$

What are options for Q?

1) $Q = \{1\}$

$\Rightarrow A \times_Q B = \{ (a,b) \in A \times B \text{ s.t. } \psi(a) = id = \chi(b) \}$

$= A \times B \cong \langle (1,2), (1,2,3), (4,5), (4,5,6) \rangle$

2) $Q = \mathrm{Sym}(3)$ i.e. $\underset{\text{id.}}{\overset{\psi}{\|}}$ and χ are isomorphisms

$\chi((4,5)) = (1,2)$
$\chi(4,5,6) = (1,2,3)$

$\Rightarrow A \times_Q B = \{ (a,b) \in A \times B \text{ s.t. } a = \chi(b) \}$

$= \langle (1,2)(4,5), (1,2,3)(4,5,6) \rangle \cong \mathrm{Sym}(3)$

3) $Q = C_2 = \langle (1,2) \rangle$ i.e. $\psi = \text{id.}$, $\chi((4,5)) = (1,2)$
$\chi((4,5,6)) = ()$.

$\Rightarrow A \times_Q B = \{ (a,b) \in A \times B \text{ s.t. } a = \chi(b) \} =$

$= \langle (1,2)(4,5), (1,2,3), (4,5,6) \rangle$