

Finding epimorphisms from f.p.  $G$  to a finite  $H$ .

If  $\varphi: G \rightarrow H$  epi,  $h \in H \Rightarrow$

$\varphi_h: G \rightarrow H$ ,  $g \mapsto h^{-1}\varphi(g)h$  is an epimorphism  
as well

$\Rightarrow$  we want to find  $\varphi$  up to an inner automorphism.

let  $G = \langle S | R \rangle$  and let  $h_i = \varphi(s_i)$ .

we need to pick the desired  $\varphi_h$ .

1) fix  $h_1 = \varphi(s_1)$  as a fixed representative from  
conjugacy class of  $h_1$ .

$\Rightarrow$  potential  $h$  belong to  $C_H(h_1) =: C_1$

2) we want to fix the conj. class of  $\varphi(s_2)$  up to elt of  $C_1$ .

let  $r$  be a representative of conjugacy class of  $h_2$ .

i.e.  $r^H = C_H(r) \setminus H$ .

act' on  $r^H$  through  $C_1 \rightsquigarrow$

this reduces the choice of  $h_2$  to a  
representative of double coset  $C_H(r) \setminus H / C_1$ .

3) we want to choose conj. class of  $\varphi(s_3)$  up to elt in  
 $C_2 := C_H(h_1, h_2)$ .

$\rightsquigarrow$  representative of double coset  
 $C_H(r) \setminus H / C_H(h_1, h_2)$ .

ALGORITHM: epimorphism

INPUT : •  $G = \langle \mathcal{S} | R \rangle$  - a f.p. group  
•  $H$  - a finite group

OUTPUT : •  $L$  - list of epimorphisms  $G \rightarrow H$

begin

$L = []$

$\text{cd}H =$  list of representatives of conjugacy classes of  $H$

$n = |\mathcal{S}|$

for  $h_1 \in \text{cd}H$  // image of  $s_1$

for  $g_2 \in \text{cd}H$  // image of  $s_2$

$\mathcal{D}_2$  - list of representatives of cosets

for  $d_2 \in \mathcal{D}_2$

$C_H(g_2) \backslash H / C_H(h_1)$

$$h_2 = g_2^{d_2}$$

⋮

for  $g_n \in \text{cd}H$

$\mathcal{D}_n$  - list of reps of cosets of

$C_H(g_n) \backslash H / C_H(h_1, \dots, h_{n-1})$

for  $d_n \in \mathcal{D}_n$

$$h_n = g_n^{d_n}$$

if is homomorphism  $(G, H, (h_1, \dots, h_n))$

$$\text{E} \langle h_1, \dots, h_n \rangle = \mathcal{G}$$

push  $(h_1, \dots, h_n)$  to  $L$

end

end

⋮

end

end

return  $L$

end

## Quotient subgroups

Defn: Let  $G$  be a f.p. group,  $H$  a (finite) group

$\varphi: G \rightarrow H$  a homomorphism.

for  $u \in H$  we say that

$$(\varphi, u) := \{g \in G : \varphi(g) \in u\} = \varphi^{-1}(u)$$

is a quotient subgroup.

Idea: compute everything in  $H$  as long as it is possible:

- to test if  $g \in \varphi^{-1}(u)$  it's enough to check if  $\varphi(g) \in u$ .

If  $\varphi$  is an epimorphism

$$G \supseteq \varphi^{-1}(u) \text{ is the same as } H \supseteq u$$

—  
this set  
is finite

←  
 $\varphi^{-1}(u)$  is of finite  
index

we can find Schreier generators

$$s_i \tau_j \overline{s_i \tau_j}^{-1} \text{ for } \varphi^{-1}(u).$$

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coset enumeration: The standard is so called

Todd-Coxeter algorithm. We'll hopefully cover it later when automata are introduced.

### Subgroup presentations

Let  $F_m \xrightarrow{\pi} G$  be a quotient homomorphism,

$$\text{i.e. } G = \langle x_1, \dots, x_m \mid r_1, \dots, r_k \rangle.$$

Let  $H < G$ ,  $[G:H] = n < \infty$ .

let  $t_1 = 1, t_2, \dots, t_n$  be a transversal for  $H \backslash G$ .

Denote by  $s_{i,j} = t_i x_j (t_i x_j)^{-1}$  the Schreier generators.

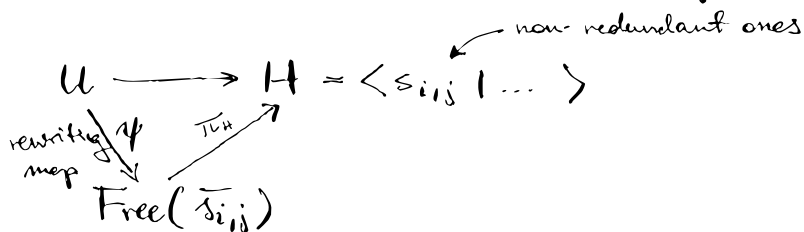
$$\begin{aligned} \text{if } t_a = t_b \cdot x_c &\Rightarrow s_{b,c} = t_b x_c (t_b x_c)^{-1} = \\ &= t_b x_c (t_a)^{-1} = 1. \end{aligned}$$

so there are at least  $n-1$  redundant generators (coming from  $t_1 = 1$ ).

Let  $U \leq F$  be  $\pi^{-1}(H)$ .

If  $w \in U \Rightarrow \pi(w) \in H$  i.e.  $\pi(w)$

can be rewritten with  $s_{i,j}$ 's.



Theorem: Let  $G = \langle S \mid R \rangle$ ,  $H < G$  of finite index.

Let  $T = \{1=t_1, \dots, t_n\}$  be a transversal for  $G/H$ .

Let  $S' = \{s_{ij} = t_i x_j (t_i x_j)^{-1} : s_{ij} \neq 1, 1 \leq i \leq n, 1 \leq j \leq m\}$

non-redundant.

•  $H = \langle S' \rangle$  (Schreier)

Let  $\pi_H: \text{Free}(S') \rightarrow H$ .

then  $\ker \pi_H = \langle \bar{t} r \bar{t}^{-1} : t \in T, r \in R \rangle$  i.e.

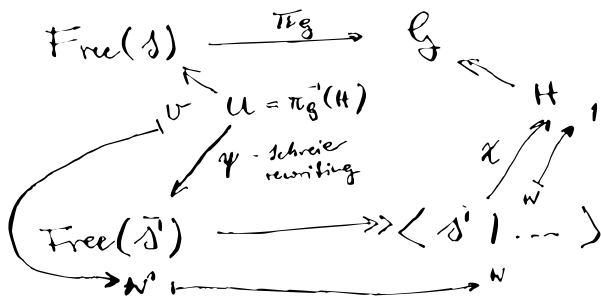
$H$  can be presented as

$\langle S' \mid \bar{t} r \bar{t}^{-1} : t \in T, r \in R \rangle$ .

Proof:

$\pi_H(\bar{t} r \bar{t}^{-1}) = 1 \Rightarrow$  the presented group maps onto  $H$ .

Sim: Any relation between  $s_{ij}$  is visible in the quotient of  $\text{Free}(S')$ .



$w'$  can be written in "old way" i.e. replacing  $\bar{s}_{ij}$  by  $t_i x_j \bar{t} x_j^{-1} \in U < \text{Free}(S)$ .

because  $\pi_g(v) = \chi(\underbrace{\pi(\psi(v))}_{w'}) = 1$

$\Rightarrow V = \prod_b r_b^{c_b}$  (product of conjugates of relations).

consider  $r^c$ ,  $r \in R$ ,  $c \in \text{Free}(S)$ .

$\pi(c)$  can be written as  $\underbrace{h \cdot t_i}_{\text{right cosets}} = t_i^{-1} \cdot h'$  for some  $t_i \in T$ ,  $h' = h^{-1} h \in H$ .

thus  $r^c = (t_i r t_i^{-1})^{\pi^{-1}(h)}$  i.e.

is an  $H$ -conjugate of something that was already trivial in  $G$ .

□

Corollary: A (finite index) subgroup of  $F_m$  is free (on  $n \cdot (m-1) - 1$  generators).

Proof: follow the procedure and since  $R = \emptyset$  there are no relations for  $H$ .

Note: we have  $n \cdot (m-1) - 1$  generators in  $S$  and  $m \cdot k$  relations. those are rather large numbers.

$\Rightarrow$  Apply Tietze transformations to shorten the presentation.

## Abelian quotients and a test for finiteness

Determine the "largest" abelian quotient.

Let  $F_m \xrightarrow{\pi} G$  be a finite presentation.

Let  $N = \langle x_i y_j^{-1} x_i^{-1} y_j^{-1} \mid x_i, y_j \in F_m \rangle < F_m$ .

then  $\pi(N) = \langle a_i b_j^{-1} a_i^{-1} b_j^{-1} \mid a_i, b_j \in G \rangle = G' \leftarrow \begin{array}{l} \text{derived} \\ \text{subgroups} \end{array}$

then  $F_m / \ker \varphi / N \cong G/G'$

$\Rightarrow G/G'$  has "abelianized" presentation for  $G$ .

write each relator of  $G$  as  $r_i = g_1^{e_{i1}} g_2^{e_{i2}} \dots g_m^{e_{im}}$

thus we obtain

a matrix:

$$R' = \begin{bmatrix} e_{11} & e_{12} & \dots & e_{1m} \\ e_{21} & e_{22} & \dots & e_{2m} \\ \vdots & & & \\ e_{n1} & e_{n2} & \dots & e_{nm} \end{bmatrix}$$

row operations on  $R'$ :

(R1)  $r_i \rightarrow r_i r_j^{\pm 1}$  - replacing one relation by a product

(R2)  $r_i \rightarrow r_i^{-1}$  - inverting a relation

column operations on  $R'$

(C1)  $g_i \rightarrow g_i g_j$  - (replace a generator by a product of two

(C2)  $g_i \rightarrow g_i^{-1}$  - invert a generator.

- Use these operations to bring  $R'$  into "Smith Normal form"

$$S = \begin{bmatrix} \alpha_1 & & 0 & & \\ & \ddots & & & \\ 0 & & \alpha_r & & 0 \\ & & & & \\ 0 & & & & 0 \end{bmatrix} \quad \begin{array}{l} \alpha_i \mid \alpha_{i+1} \text{ for } 1 \leq i \leq r \\ \alpha_i \geq 1 \end{array}$$

$$\Rightarrow G/G' \cong \bigoplus_{i=1}^r C_{\alpha_i} \oplus \mathbb{Z}^{\beta_g} \quad (\beta = \min(k-r, m-r))$$

Corollary: • if  $\beta_g > 0 \Rightarrow G$  is infinite.

• the same is true if for  $H < G$   
(of finite index, e.g. given by a quotient subgroup)

$$\beta_H \text{ is } > 0.$$

General procedure:

- find a subgroup  $H < G$  of small index
- determine  $\beta_H$
- if  $\beta_H = 0$  repeat with different  $H$ .