Finding epimorphisms from f.p. G to a finite H.

If $\varphi: G \to H$ epi, $h \in H$ $\Rightarrow$

$\varphi_h: G \to H$, $g \mapsto h^{-1}\varphi(g)h$ is an epimorphism as well

$\Rightarrow$ we want to find φ up to an inner automorphism.

let $G = \langle S \mid R \rangle$ and let $h_i = \varphi(s_i)$.

we need to pick the desired φ_h.

1) fix $h_1 = \varphi(s_1)$ as a fixed representative from conjugacy class of h_1.

$\Rightarrow$ potential h belong to $C_H(h_1) =: C_1$

2) we want to fix the conj. class of $\varphi(s_2)$ up to elt of C_1.

let r be a representative of conjugacy class of h_2.

i.e. $r^H = C_H(r) \backslash H$.

act' on r^H through C_1 $\rightsquigarrow$

this reduces the choice of h_2 to a representative of double coset $C_H(r) \backslash H / C_1$.

3) we want to choose conj. class of $\varphi(s_3)$ up to elt in $C_2 := C_H(h_1, h_2)$.

$\rightsquigarrow$ representative of double coset $C_H(r) \backslash H / C_H(h_1, h_2)$.

ALGORITHM: epimorphism

INPUT : • $G = \langle S \mid R \rangle$ – a f. p. group
• H – a finite group

OUTPUT : • L – list of epimorphisms $G \to H$

begin
$L = [\,]$
ccl$H$ = list of representatives of conjugacy classes of H
$n = |S|$
for h_1 in ccl$H$ // image of s_1
for $g_2 \in$ ccl$H$ // image of s_2
D_2 – list of representatives of cosets $C_H(g_2)\backslash H / C_H(h_1)$
for $d_2 \in D_2$
$h_2 = g_2^{d_2}$
⋮
for $g_n \in$ ccl$H$
D_n – list of reps of cosets of $C_H(g_n)\backslash H / C_H(h_1, \dots, h_{n-1})$
for $d_n \in D_n$
$h_n = g_n^{d_n}$
if ishomomorphism$(G, H, (h_1, \dots, h_n))$ & $\langle h_1, \dots, h_n \rangle = G$
push $(h_1, \dots, h_n)$ to L
end
end
⋮
end
end
end
return L
end

Quotient subgroups

Defn: Let G be a f.p. group, H a (finite) group $\varphi : G \to H$ a homomorphism.

For $U \leq H$ we say that

$$(\varphi, U) := \{ g \in G : \varphi(g) \in U \} = \varphi^{-1}(U)$$

is a quotient subgroup.

Idea: compute everything in H as long as it is possible:

- to test if $g \in \varphi^{-1}(U)$ it's enough to check if $\varphi(g) \in U$.

If φ is an epimorphism

$G \curvearrowright {}_{\varphi^{-1}(U)}\backslash G$ is the same as $H \curvearrowright {}_{U}\backslash H$.

this set is finite

$\Longleftarrow$

$\varphi^{-1}(U)$ is of finite index

we can find Schreier generators

$s_i r_j \overline{s_i r_j}^{-1}$ for $\varphi^{-1}(U)$.

Coset enumeration: The standard is so called Todd–Coxeter algorithm. We'll hopefully cover it later when Automata are introduced.

Subgroup presentations

Let $F_m \xrightarrow{\pi} G$ be a quotient homomorphism,

i.e. $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_k \rangle$.

Let $H < G$, $[G:H] = n < \infty$.

let $t_1 = 1, t_2, \dots, t_n$ be a transversal for $H \backslash G$.

Denote by $s_{i,j} = t_i x_j \left(\overline{t_i x_j}\right)^{-1}$ the Schreier generators.

$$\text{if } t_a = t_b \cdot x_c \Rightarrow s_{b,c} = t_b x_c \left(\overline{t_b x_c}\right)^{-1} = t_b x_c (t_a)^{-1} = 1.$$

so there are at least $n-1$ redundant generators (coming from $t_1 = 1$).

Let $U \leq F$ be $\pi^{-1}(H)$.

If $w \in U \Rightarrow \pi(w) \in H$ i.e. $\pi(w)$ can be rewritten with $s_{i,j}$'s.

non-redundant ones

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "\psi", "\text{rewriting map}"'] & H = \langle s_{i,j} \mid \dots \rangle \\
\mathrm{Free}(s_{i,j}) \arrow[ur, "\pi|_H"'] &
\end{tikzcd}$$

Theorem: Let $G = \langle S \mid R \rangle$, $S = \{x_1, \dots, x_m\}$, $H < G$ of finite index.

Let $T = \{1 = t_1, \dots, t_n\}$ be a transversal for $H \backslash G$.

Let $S' = \left\{ s_{i,j} = t_i x_j \left(\overline{t_i x_j}\right)^{-1} : s_{i,j} \neq 1,\ 1 \le i \le n,\ 1 \le j \le m \right\}$

non-redundant.

- $H = \langle S' \rangle$ (Schreier)

Let $\pi_H : \mathrm{Free}(\bar{S}') \longrightarrow H$.

then $\ker \pi_H = \langle \bar{t}\, \bar{r}\, \bar{t}^{-1} : t \in T, r \in R \rangle$ i.e.

sloppy

H can be presented as

$$\langle \bar{S}' \mid \bar{t}\, \bar{r}\, \bar{t}^{-1} : t \in T, r \in R \rangle.$$

Proof:

$\pi_H(\bar{t}\, \bar{r}\, \bar{t}^{-1}) = 1 \implies$ the presented group maps onto H.

Aim: Any relation between $s_{i,j}$ is visible in the quotient of $\mathrm{Free}(\bar{S}')$.

$$\begin{tikzcd}
\mathrm{Free}(S) \arrow[rr, "\pi_G"] & & G \\
U = \pi_G^{-1}(H) \arrow[u, hook] \arrow[d, "\psi\ \text{- Schreier rewriting}"'] & & H \arrow[u, hook] \\
\mathrm{Free}(\bar{S}') \arrow[rr, two heads] & & \langle S' \mid \dots \rangle \arrow[u, "\chi"]
\end{tikzcd}$$

$$v \mapsto w' \mapsto w \mapsto 1$$

w' can be written in "old way" i.e. replacing $\bar{s}_{ij}$ by $t_i x_j \overline{t_i x_j}^{-1} \in U < \mathrm{Free}(S)$.

because $\pi_g(v) = \chi(\pi(\underbrace{\underbrace{\psi(v)}_{w'})}_{w'}) = 1$

$\Rightarrow v = \prod_b r_b^{c_b}$ (product of conjugates of relations).

consider r^c, $r \in R$, $c \in \mathrm{Free}(S)$.

$\pi(c)$ can be written as $\underline{h \cdot t_i} = t_i^{-1} h'$ for some $t_i \in T$

right cosets. $h' = h^{-1}$, $h \in H$.

thus $r^c = \left(t_i \, r \, t_i^{-1}\right)^{\pi^{-1}(h)}$ i.e.

is an H-conjugate of something that was already trivial in G.

$\square$

<u>Corollary</u>: $\forall$ (finite index $= n$) subgroup of F_m is free (on $n \cdot (m-1) - 1$ generators).

<u>Proof</u>: follow the procedure and since $R = \emptyset$ there are no relations for H.

<u>Note</u>: we have $n \cdot (m-1) - 1$ generators in S and $m \cdot k$ relations. those are rather large numbers.

$\Rightarrow$ Apply Tietze transformations to shorten the presentation.

Abelian quotients and a test for finiteness.

Determine the "largest" abelian quotient.

Let $F_m \xrightarrow{\pi} G$ be a finite presentation.

Let $N = \langle x^{-1}y^{-1}xy \mid x, y \in F_m \rangle < F_m$.

then $\pi(N) = \langle a^{-1}b^{-1}ab \mid a, b \in G \rangle = G'$ ← derived subgroups

$$\text{then } F_m/\ker\varphi/N \cong G/G'$$

$\Rightarrow$ G/G' has "abelianized" presentation for G.

write each relator of G as $r_i = g_1^{e_{i1}} g_2^{e_{i2}} \dots g_m^{e_{im}}$

thus we obtain a matrix :

$$R' = \begin{bmatrix} e_{11} & e_{12} & \dots & e_{1m} \\ e_{21} & e_{22} & \dots & e_{2m} \\ \vdots & & & \\ e_{n1} & e_{n2} & \dots & e_{nm} \end{bmatrix}$$

row operations on R':

(R1) $r_i \to r_i r_j^{\pm 1}$ – replacing one relation by a product of two

(R2) $r_i \to r_i^{-1}$ – inverting a relation

column operations on R'

(C1) $g_i \to g_i g_j$ – (replace a generator by a product of two

(C2) $g_i \to g_i^{-1}$ – invert a generator.

- Use these operations to bring R' into "Smith Normal form"

$$S = \begin{bmatrix} \alpha_1 & \ddots & 0 & \\ 0 & & \alpha_r & 0 \\ 0 & & & 0 \end{bmatrix} \qquad \begin{array}{l} \alpha_i \mid \alpha_{i+1} \text{ for } 1 \le i \le r \\ \alpha_i \ge 1 \end{array}$$

$$\Rightarrow \quad G/G' \cong \bigoplus_{i=1}^{r} C_{\alpha_i} \oplus \mathbb{Z}^{\beta_G} \qquad \beta = \min(k-r, m-r)$$

<u>Corollary:</u> • if $\beta_G > 0 \Rightarrow G$ is infinite.

- the same is true if for $H < G$ (of finite index, eg. given by a quotient subgroup) β_H is > 0.

<u>General procedure:</u>

- find a subgroup $H < G$ of small index
- determine β_H
- If $\beta_H = 0$ repeat with different H.