## Free Groups:

Let $S$ be a set.

If $F$ is a group and $S \subset F$ we say that $S$ <u>freely generates</u> $F$ iff $\forall\, \varphi : S \to G$

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "id"'] & G \\ F \arrow[ur, "\exists!\, \overline{\varphi}"'] & \end{tikzcd}$$

(every map $S \xrightarrow{\varphi} G$ can be uniquely extended to a homomorphism $\overline{\varphi}$).

We say that <u>$F$ is free</u> iff $F$ is freely generated by a set.

<u>Ex:</u>
- $\mathbb{Z}$ is freely generated by $\{1\}$.
- $\mathbb{Z}$ is <u>not</u> freely generated by $\{2,3\}$.
  (but it is generated by $\{2,3\}$).
- $\mathbb{Z}/2\mathbb{Z}$ is not free
- $\mathbb{Z}^2$ is not free.

Proposition: Let $S$ be a set.

There is (up to the canonical isomorphism) at most one free group generated by $S$.

(i.e. the universal property of $\mathrm{Free}(S)$).

1) If $F$ and $F'$ are the universal objects w.r.t. $S$, then

$$
\begin{tikzcd}
S \arrow[r, hook, "\varphi"] \arrow[d, "\varphi'"'] & F \\
F' \arrow[ur, "\overline{\varphi}"'] &
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
S \arrow[r, hook, "\varphi'"] \arrow[d, "\varphi"'] & F' \\
F \arrow[ur, "\overline{\varphi'}"'] &
\end{tikzcd}
$$

these are commutative. so we have

$$\varphi' = \overline{\varphi'} \circ \varphi \quad \& \quad \overline{\varphi} \circ \varphi' = \varphi$$

Aim: $\overline{\varphi} \circ \overline{\varphi'} : F \to F$ is $id_F$.

$$
\begin{tikzcd}
S \arrow[r, hook, "\varphi"] \arrow[d, "\varphi"'] & F \\
F \arrow[ur, "\psi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[rr, "\varphi"] \arrow[d, "\varphi"'] \arrow[dr, "\varphi'"] & & F \\
F \arrow[r, "\overline{\varphi'}"'] & F' \arrow[ur, "\overline{\varphi}"'] &
\end{tikzcd}
$$

$\psi \circ \varphi = \varphi$ one such $\psi$ is $id_F$.

but we could choose $\psi = \overline{\varphi} \circ \overline{\varphi'}$ as well:

$$\overline{\varphi} \circ \underbrace{\overline{\varphi'} \circ \varphi} = \underbrace{\overline{\varphi} \circ \varphi'} = \varphi.$$

By the uniqueness $\overline{\varphi} \circ \overline{\varphi'} = id_F$

$\Rightarrow$ (the same with $F'$) $\Rightarrow F \cong F'$ $\square$

Theorem:

Let $S$ be a set. Then there exists

Free$(S)$ – the free group generated by $S$.

Proof:

$A = \{S \cup \hat{S}\}$ ← $\hat{S}$ the set of formal inverses of elts from $S$.

↖ alphabet

1) $A^*$ – the set of all (including the empty one) words over the alphabet $A$.

$$\bullet : A^* \times A^* \longrightarrow A^*$$

$$(w_1, w_2) \longmapsto w_1 w_2 \quad \text{(word concatenation).}$$

it is associative and $\varepsilon$ (the empty word) is the neutral element.

2) $\sim \subset A^* \times A^*$ – a relation generated by:

$$\forall x, y \in A^* \ \forall s \in S \ (x s \hat{s} y, xy)$$

$$\text{— " —} \quad (x \hat{s} s y, xy).$$

??? the smallest eq. relation which contains all of these.

$$\text{Free}(S) = F(S) := A^* \times A^* / \sim$$

$$w_\sim = [w]$$

$$[a] \cdot [b] = [a \cdot b]$$

Check that this is well defined, associative and $[\varepsilon]$ is the neutral element.

## The existence of inverses:

$$[\varepsilon]^{-1} = [\varepsilon]$$

$$\left.\begin{array}{lll} [sx]^{-1} = [x]^{-1}\cdot[\hat{s}] & \forall x \in \mathcal{A}^* \\ [\hat{s}x]^{-1} = [x]^{-1}\cdot[s] & \forall x \in \mathcal{A}^* \end{array}\right\} \text{inductive definition}$$

By induction:

$$\left([sx]^{-1}\right)^{-1} = \left([x]^{-1}\cdot[\hat{s}]\right)^{-1} = \left([\hat{s}]^{-1}\cdot\underbrace{\left([x]^{-1}\right)^{-1}}_{=[x] \text{ by induction}}\right) =$$

$$= [s]\cdot[x] = [sx]$$

$$[sx]^{-1}\cdot[sx] = [x]^{-1}\cdot[\hat{s}]\cdot[s]\cdot[x] =$$

$$= [x]^{-1}\cdot[\hat{s}\cdot s]\cdot[x] = \underbrace{[x]^{-1}\cdot[x] = \varepsilon}_{\text{by induction.}}$$

$\Rightarrow$ Free($\mathcal{S}$) is a group.

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<u>Universality:</u> let $i: S \hookrightarrow \text{Free}(\mathcal{S})$

$$s \longmapsto [s].$$

By construction every $[x] \in$ Free can be written as a word in $[s]$es and $[\hat{s}]$es.

$\Rightarrow$ $i(\mathcal{S}) \subset \text{Free}(\mathcal{S})$ generates (as a group!).

- $i$ is injective
- every map $S \xrightarrow{\varphi} G$ extends to a homomorphism

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "i"'] & G \\ \mathrm{Free}(S) \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}.$$

How to define $\overline{\varphi}$? we start with $\varphi^*: \mathcal{A}^* \to G$

$$\varphi^*(\varepsilon) = \mathbb{1}_G$$
$$\varphi^*(s) = \varphi(s)$$
$$\varphi^*(\hat{s}) = (\varphi(s))^{-1}$$
$$\left.\begin{aligned} \varphi^*(sx) &= \varphi(s)\cdot\varphi^*(x) \\ \varphi^*(\hat{s}x) &= \varphi^*(\hat{s})\cdot\varphi^*(x) \end{aligned}\right\} + \text{induction.}$$

we want to say $\overline{\varphi}([x]) = \varphi^*(x)$.

note that
$$\begin{array}{ccc} \overline{\varphi}([x]) & = & \overline{\varphi}([s\hat{s}x]) \\ \| & & \| \\ \varphi^*(x) & & \varphi^*(s\hat{s}x) \\ & & \| \\ & & \varphi^*(s)\cdot\varphi(s)^{-1}\cdot\varphi^*(x) \\ & & \| \\ & & \varphi^*(x) \end{array}$$

$\Rightarrow$ $\varphi^*$ is compatible with $\sim$ on $\mathcal{A}^*$

constant on equivalence classes

$\Longrightarrow$ $\overline{\varphi}: \mathrm{Free}(S) \to G$
$\overline{\varphi}([x]) = \varphi^*(x)$
is well defined.

$i : S \longrightarrow \mathrm{Free}(S)$ is injective.

consider $\varphi : S \to \mathbb{Z}$

$$\varphi(s_1) = 1$$

$$\varphi(s_2) = -1.$$

then $\overline{\varphi}(i(s_1)) = \overline{\varphi}([s_1]) = \varphi^*(s_1) = 1$

$\overline{\varphi}(i(s_2)) = \quad \cdots \quad = -1$

$\Rightarrow \quad i(s_1) \neq i(s_2).$

$\square$

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<u>Defn</u>: Rank.

Let $F$ be a free group. If $S$ generates $F$ freely, then $|S|$ is called <u>the rank of $F$.</u>

<u>Proposition</u>: The rank of $F$ is well defined.

<u>Proof</u>: Let $S$ be a free generating set and let $S'$ be any generating set for $F$. We'll show that $|S| \leq |S'|$.

$$\begin{tikzcd} S \arrow[r, "\varphi_i"] \arrow[d, hook] & \{\pm 1\} \cong C_2 \\ S' \subset F \arrow[ur, "\overline{\varphi}_i"'] & \end{tikzcd}$$

$|S|$ different homoms.

$$\varphi_i(s_j) = \begin{cases} -1 & \text{iff } i = j \\ 1 & \text{otherwise} \end{cases}$$

If $F = \langle S' \rangle$ there are at most $|S'|$ different ones.

Warning: $F_2$ has subgroups isomorphic to $F_n$ for any $n$ including $\infty$!.

$$F_n := \mathrm{Free}(\{x_1, \dots, x_n\}) := \langle x_1, \dots, x_n \rangle$$

Corollary:

A group is finitely generated iff it is a quotient of a free group, i.e. $G$ is f.g on $n$ generators, iff there exists an epimorphism $F_n \twoheadrightarrow G$.

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Defn: Let $S \subset G$. then

- $\langle S \rangle \leq G$ is the subgroup generated by elements from $S$.
- $\langle\langle S \rangle\rangle := N_G(\langle S \rangle)$ is the "normal closure" of $S$, i.e. the smallest normal subgroup of $G$ that contains $S$.
- We will write $G = \langle s_1, \dots, s_n \mid r_1, \dots, r_k \rangle$, where $S = \{s_1, \dots, s_n\}$ and $r_i \in \mathrm{Free}(S)$ to denote

$$\mathrm{Free}(S) / \langle\langle r_1, \dots, r_k \rangle\rangle .$$

we say that $G$ is generated by $S$ subject to relations $r_1, \dots, r_k$.

- $S = \{s_1, \dots, s_m\}$, $R = \{r_1, \dots, r_k\}$

$\Rightarrow$ $\langle S \mid R \rangle$ is <u>a</u> <u>presentation</u> for $G$

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<u>Proposition</u>: Let $S$ be a set and $R \subset \mathcal{A}(S)^*$ be a set of words. Let $\pi$ be the unique map

$$\begin{tikzcd} S \arrow[r, "s \mapsto {[s]}"] \arrow[d] & \langle S \mid R \rangle \\ F(S) \arrow[ur, "\pi"'] & \end{tikzcd}$$

Let

$$\begin{tikzcd} & S \arrow[dl] \arrow[d] \arrow[r, dashed, "\varphi"] & G \\ F(S) \arrow[r, "\pi"'] & \langle S \mid R \rangle \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

be a map s.t. $\varphi^*(r) = \mathbb{1}_G \quad \forall r \in R$.

Then there exists a unique homomorphism $\overline{\varphi}$ s.t. $\varphi = \overline{\varphi} \circ \pi$.

$\square$

<u>Examples</u>: $\langle x \mid x^n \rangle \cong C_n$, $S = \{x\}$, $R = \{x^n\}$

$$\begin{tikzcd} & \{x\} \arrow[dl] \arrow[d] \arrow[r, "\varphi"] & C_n = \left(\{\zeta_n^i,\ i = 0, \dots, n-1\}, \cdot, \zeta_n^0 = 1\right) \\ F(\{x\}) \arrow[r, "\pi"'] & \langle x \mid x^n \rangle \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

$\varphi(x) = \zeta_n$

$\varphi(x^n) = \zeta_n^n = 1$

- $\langle x,y \mid x^{-1}y^{-1}xy \rangle \cong \mathbb{Z}^2$

$$
\begin{tikzcd}
 & \{x,y\} \arrow[dl] \arrow[dr, "i"] \arrow[rr, "\varphi"] & & \mathbb{Z}^2 \\
F(x,y) \arrow[rr, "\pi"] & & \langle x,y \mid x^{-1}y^{-1}xy \rangle \arrow[ur, dotted, "\bar{\varphi}"'] &
\end{tikzcd}
$$

$\varphi(x) = (1,0)$
$\varphi(y) = (0,1)$

$\varphi = \bar{\varphi} \circ \pi$

$$
\begin{tikzcd}
 & \{x,y\} \arrow[dl] \arrow[dr, "\varphi"] \arrow[rr, "i"] & & \langle x,y \mid x^{-1}y^{-1}xy \rangle & \mathcal{G} \arrow[l] \\
F(x,y) \arrow[rr, "\pi'"] & & \mathbb{Z}^2 \arrow[ur, dotted, "\bar{i}"'] & &
\end{tikzcd}
$$

$$i = \bar{i} \circ \pi'$$

$$F(x,y) \xrightarrow{\pi} \mathcal{G} \xrightarrow{\bar{\varphi}} \mathbb{Z}^2 \xrightarrow{\bar{i}} \mathcal{G}$$

$$\bar{i} \circ \underline{\bar{\varphi} \circ \pi} = \bar{i} \circ \varphi = \bar{i} \circ \pi' = i \;\Big|\; \text{on the generating set.}$$

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<u>Defn</u>: A <u>finitely presented group</u> is a group isomorphic to a quotient $F_n / \langle\langle R \rangle\rangle$ for finite $n$ and $R$.

<u>Note</u>: there are many presentations of the same group $\to$ examples?

Fun fact: a finite group $G$ is f.p.

Proof: let $n = |G|$ and $\varphi : F_n \to G$.

$\ker\varphi \trianglelefteq F_n$ is of index $n$.

pick a set of representatives for $\ker\varphi \backslash F_n$

$r_1, \dots, r_k$.

Then $\ker\varphi = \underbrace{\langle x_i r_j (\overline{x_i r_j})^{-1} \rangle_{\substack{i=1,\dots,n \\ k=1,\dots,k}}}_{\text{finite!}}$

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Working with presentations:

Theorem (Tietze)

Let $G = \langle S | R \rangle$ and let $\langle S' | R' \rangle$ be the result of one of the following transformations.

Then $\langle S' | R' \rangle \cong G$.

(T1) $R' = R \cup \{r\}$, $r \in A(R)^*$

(T2) $R' = R \setminus \{r\}$, $r \in A(R \setminus \{r\})^*$

(T3) $S' = S \cup \{x\}$, $R' = R \cup \{x^{-1} w\}$ for any $w \in A(S)^*$

(T4) $S' = S \setminus \{x\}$, $R' = R \setminus \{r\}$ when $x$ occurs only once in $r$ and does not occur in $R'$.

$\square$

Suppose that $\langle S_1 | R_1 \rangle \cong \langle S_2 | R_2 \rangle$.

__Theorem (Tietze)__: there is a word $w \in \mathcal{A}(\{T1, \dots, T4\})$ which transforms $\langle S_1 | R_1 \rangle$ to $\langle S_2 | R_2 \rangle$.

__Proof:__ If $\varphi : \langle S_1 | R_1 \rangle \longrightarrow \langle S_2 | R_2 \rangle$ is an isomorphism, then

$$\varphi(s) \text{ is a word in } \mathcal{A}(S_2)^*.$$

$$\begin{tikzcd}
\langle S_2 | R_2 \rangle \arrow[r, "T3s"] & \langle S_1 \cup S_2 | R_2 \cup \underbrace{\{s^{-1}\varphi(s) : s \in S_1\}}_{R_3} \rangle \arrow[d, "T1s"] \\
\langle S_1 \cup S_2 | R_1 \cup R_3 \rangle \arrow[d, "T4s"] & \langle S_1 \cup S_2 | R_1 \cup R_2 \cup R_3 \rangle \arrow[l, "T2s"'] \\
\langle S_1 | R_1 \rangle .
\end{tikzcd}$$

$\square$

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Again with this "algorithm" we face a problem of graph exploration. However there are good heuristics to locally "minimize" a presentation: only look for "local minima".

More useful transformations (human readable):

(T1'): replace $r$ by its (minimal) cyclicly reduced form.

(T2'): if $r_1 = abc$, $r_2 = dbf$ $\quad (a,b,c,d,f \in \mathcal{A}(S)^*)$

$$\Rightarrow b = d^{-1}f^{-1} \Rightarrow r_1' = ad^{-1}f^{-1}c \text{ \& remove } r_2$$

(T3'): If $r = axb$, $a, b \in \mathcal{A}(S\setminus\{x\})^*$

$\Rightarrow x = a^{-1}b^{-1} \Rightarrow$ remove $x$ from $S$ & replace occurences of $x$ with $a^{-1}b^{-1}$.

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In practice more than just local minima are explored:

1) eliminate gens using rels of length 1 or 2
2) eliminate some generators using (T3') at the cost of increasing the presentation length
3) find common intersections of relators to eliminate them using (T2').