

## Free Groups:

Let  $S$  be a set.

If  $F$  is a group and  $S \subset F$  we say that

$S$  freely generates  $F$  iff  $\forall \varphi: S \rightarrow G$

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & G \\ \downarrow \cong & \nearrow & \\ F & \xrightarrow{\exists! \bar{\varphi}} & \end{array} \quad (\text{every map } S \rightarrow G \text{ can be uniquely extended to a homomorphism } \bar{\varphi}).$$

We say that  $F$  is free iff  $F$  is freely generated by a set.

- Ex:
- $\mathbb{Z}$  is freely generated by  $\{1\}$ .
  - $\mathbb{Z}$  is not freely generated by  $\{2, 3\}$ .  
(but it is generated by  $\{2, 3\}$ ).
  - $\mathbb{Z}/2\mathbb{Z}$  is not free
  - $\mathbb{Z}^2$  is not free.

Proposition: Let  $S$  be a set.

There is (up to the canonical isomorphism) at most one free group generated by  $S$ .

(i.e. the universal property of  $\text{Free}(S)$ ).

1) If  $F$  and  $F'$  are the universal objects v.r.t.  $S$ , then

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & F \\ \varphi' \downarrow & \nearrow \bar{\varphi} & \\ F' & & \end{array} \quad \& \quad \begin{array}{ccc} S & \xrightarrow{\varphi'} & F' \\ \varphi \downarrow & \nearrow \bar{\varphi}' & \\ F & & \end{array}$$

these are commutative. so we have

$$\varphi' = \bar{\varphi}' \circ \varphi \quad \& \quad \bar{\varphi} \circ \varphi' = \varphi$$

Sim:  $\bar{\varphi} \circ \bar{\varphi}' : F \rightarrow F$  is  $\text{id}_F$ .

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & F \\ \varphi' \downarrow & \nearrow \psi & \\ F & & \end{array} \quad \begin{array}{ccc} S & \xrightarrow{\varphi} & F \\ \varphi' \downarrow & \searrow \varphi' & \nearrow \bar{\varphi} \\ F & \xrightarrow{\bar{\varphi}'} & F' \end{array}$$

$\psi \circ \varphi = \varphi$  one such  $\psi$  is  $\text{id}_F$ .

but we could choose  $\psi = \bar{\varphi} \circ \bar{\varphi}'$  as well:

$$\bar{\varphi} \circ \underbrace{\bar{\varphi}' \circ \varphi} = \bar{\varphi} \circ \underbrace{\varphi'} = \varphi.$$

By the uniqueness  $\bar{\varphi} \circ \bar{\varphi}' = \text{id}_F$   
 $\Rightarrow$  (the same with  $F'$ )  $\Rightarrow F \cong F'$   $\square$

Theorem:

Let  $S$  be a set. Then there exists

$\text{Free}(S)$  - the free group generated by  $S$ .

Proof:

$A = \{S \cup \hat{S}\}$  ←  $\hat{S}$  the set of formal inverses of elts from  $S$ .  
↑  
alphabet

1)  $A^*$  - the set of all (including the empty one) words over the alphabet  $A$ .

$$\cdot : A^* \times A^* \rightarrow A^*$$

$$(w_1, w_2) \mapsto w_1 w_2 \text{ (word concatenation).}$$

it is associative and  $\underline{\varepsilon}$  (the empty word) is the neutral element.

2)  $\sim \subset A^* \times A^*$  - a relation generated by:

$$\forall x, y \in A^* \forall s \in S (x s \hat{s} y, x y)$$

$$\text{-----} (x \hat{s} s y, x y).$$

↓  
???  
the smallest eq. relation which contains all of these.

$$\text{Free}(S) = F(S) := A^* \times A^* / \sim$$

$$w_{\sim} = [w]$$

$$[a] \cdot [b] = [a \cdot b]$$

Check that this is well defined, associative and  $[\varepsilon]$  is the neutral element.

The existence of inverses:

$$[\varepsilon]^{-1} = [\varepsilon]$$

$$[sx]^{-1} = [x]^{-1} \cdot [\hat{s}] \quad \forall x \in \mathcal{A}^*$$

$$[\bar{s}x]^{-1} = [x]^{-1} \cdot [s] \quad \forall x \in \mathcal{A}^*$$

} inductive definition

By induction:

$$\begin{aligned} ([sx]^{-1})^{-1} &= ([x]^{-1} \cdot [\hat{s}])^{-1} = \left( [\hat{s}]^{-1} \cdot ([x]^{-1})^{-1} \right) = \\ &= [s] \cdot [x] = [sx] \end{aligned}$$

= [x] by induction

$$\begin{aligned} [\bar{s}x]^{-1} \cdot [sx] &= [x]^{-1} \cdot [\bar{s}] \cdot [s] \cdot [x] = \\ &= [x]^{-1} \cdot [\bar{s} \cdot s] \cdot [x] = [x]^{-1} \cdot [x] = \varepsilon \end{aligned}$$

by induction.

$\Rightarrow$  Free( $S$ ) is a group.

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Universality: let  $i: S \hookrightarrow \text{Free}(S)$   
 $s \mapsto [s]$ .

By construction every  $[x] \in \text{Free}$  can be written as a word in  $[s]_s$  and  $[\hat{s}]_s$ .

$\Rightarrow i(S) \subset \text{Free}(S)$  generates (as a group!).

- $i$  is injective

- every map  $S \xrightarrow{\varphi} G$  extends to

a homomorphism

$$\begin{array}{ccc} & & G \\ & \nearrow \bar{\varphi} & \\ \text{Free}(S) & & \end{array}$$

How to define  $\bar{\varphi}$ ? we start with  $\varphi^*: A^* \rightarrow G$

$$\varphi^*(\varepsilon) = 1_G$$

$$\varphi^*(s) = \varphi(s)$$

$$\varphi^*(s^{-1}) = (\varphi(s))^{-1}$$

$$\varphi^*(sx) = \varphi(s) \cdot \varphi^*(x)$$

$$\varphi^*(s^{-1}x) = \varphi^*(s)^{-1} \cdot \varphi^*(x)$$

} + induction.

we want to say  $\bar{\varphi}([x]) = \varphi^*(x)$ .

note that  $\bar{\varphi}([x]) = \bar{\varphi}([s\hat{s}x])$

$$\downarrow$$

$$\varphi^*(x)$$

$$\downarrow$$

$$\varphi^*(s\hat{s}x)$$

$$\downarrow$$

$$\varphi^*(s) \cdot \varphi^*(\hat{s})^{-1} \cdot \varphi^*(x)$$

$$\downarrow$$

$$\varphi^*(x)$$

$\Rightarrow \varphi^*$  is compatible with

$\sim$  on  $A^*$

$\Rightarrow$  constant on equivalence classes

$$\bar{\varphi}: \text{Free}(S) \rightarrow G$$

$$\bar{\varphi}([x]) = \varphi^*(x)$$

is well defined.

$i: S \rightarrow \text{Free}(S)$  is injective.

consider  $\varphi: S \rightarrow \mathbb{Z}$ ,

$$\varphi(s_1) = 1$$

$$\varphi(s_2) = -1.$$

then  $\overline{\varphi}(i(s_1)) = \overline{\varphi}([s_1]) = \varphi^*(s_1) = 1$   
 $\overline{\varphi}(i(s_2)) = \dots = -1$

$\Rightarrow i(s_1) \neq i(s_2).$

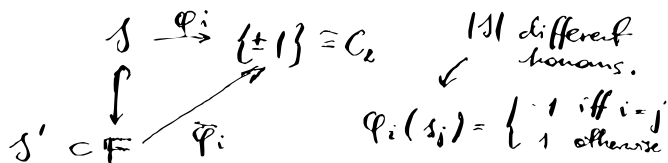


Defn: Rank.

Let  $F$  be a free group. If  $S$  generates  $F$  freely, then  $|S|$  is called the rank of  $F$ .

Proposition: The rank of  $F$  is well defined.

Proof: Let  $S$  be a free generating set and let  $S'$  be any generating set for  $F$ . We'll show that  $|S| \leq |S'|$ .



If  $F = \langle S' \rangle$  there are at most  $|S'|$  different ones.

Warning:  $F_2$  has subgroups isomorphic to  $F_n$  for any  $n$  including  $\infty$ !

$$F_n := \text{Free}(\{x_1, \dots, x_n\}) := \langle x_1, \dots, x_n \rangle$$

Corollary:

△ group is finitely generated iff it is a quotient of a free group, i.e.  $G$  is f.g. on  $n$  generators, iff there exists an epimorphism  $F_n \rightarrow G$ .

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Defn: Let  $S \subset G$ . then

- $\langle S \rangle \leq G$  is the subgroup generated by elements from  $S$ .
- $\langle\langle S \rangle\rangle := N_G(\langle S \rangle)$  is the "normal closure" of  $S$ , i.e. the smallest normal subgroup of  $G$  that contains  $S$ .
- We will write  $G = \langle s_1, \dots, s_n \mid r_1, \dots, r_k \rangle$ , where  $S = \{s_1, \dots, s_n\}$  and  $r_i \in \text{Free}(S)$  to denote

$$\text{Free}(S) / \langle\langle r_1, \dots, r_k \rangle\rangle.$$

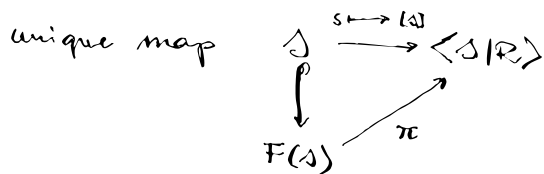
we say that  $G$  is generated by  $S$  subject to relations  $r_1, \dots, r_k$ .

$$\cdot \mathcal{S} = \{s_1, \dots, s_m\}, \mathcal{R} = \{r_1, \dots, r_k\}$$

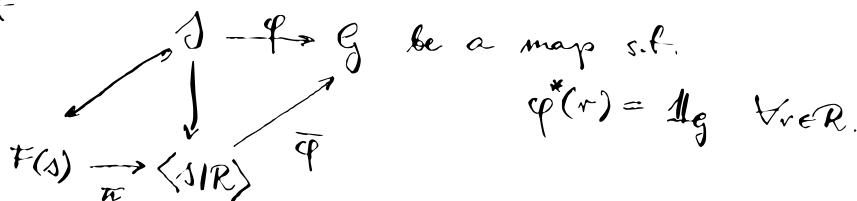
$\Rightarrow \langle \mathcal{S} | \mathcal{R} \rangle$  is a presentation for  $G$

Proposition: Let  $\mathcal{S}$  be a set and  $\mathcal{R} \subset \mathcal{A}(\mathcal{S})^*$

be a set of words. Let  $\pi$  be the



Let



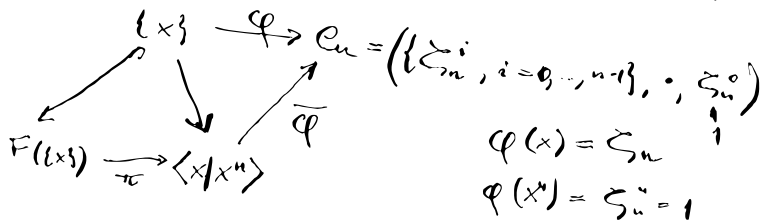
then there exists a unique homomorphism  $\bar{\varphi}$

s.t.  $\varphi = \bar{\varphi} \circ \pi$ .

□

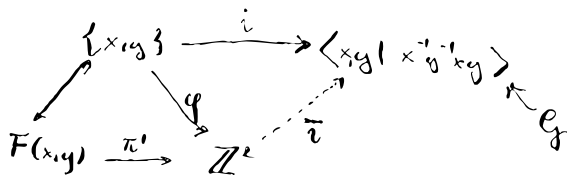
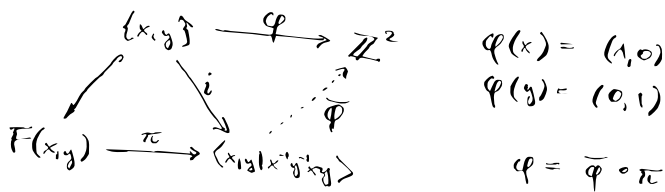
Examples:

$$\langle x | x^n \rangle \cong C_n, \quad \mathcal{S} = \{x\}, \mathcal{R} = \{x^n\}$$





•  $\langle x, y \mid x^2, y^2 \rangle \cong \mathbb{Z}^2$



$$F(x, y) \xrightarrow{\pi} G \xrightarrow{\bar{\varphi}} \mathbb{Z}^2 \xrightarrow{\bar{i}} G$$

$$\bar{i} \circ \bar{\varphi} \circ \pi = \bar{i} \circ \varphi = \bar{i} \circ \pi' = i \quad (\text{on the generating set.})$$

Defn: A finitely-presented group is a group isomorphic to a quotient  $F_n / \langle\langle R \rangle\rangle$  for finite  $n$  and  $R$ .

Note: there are many presentations of the same group  $\rightarrow$  examples?

Fun fact: a finite group  $G$  is f.p.

Proof: let  $n = |G|$  and  $\varphi: F_n \rightarrow G$ .

$\ker \varphi \triangleleft F_n$  is of index  $n$ .

pick a set of representatives for  $\ker \varphi \backslash F_n$

$x_1, \dots, x_k$ .

Then  $\ker \varphi = \langle \underbrace{x_i r_j (x_i r_j)^{-1}}_{\substack{i=1, \dots, n \\ j=1, \dots, k}} \rangle$   
finite!

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Working with presentations:

Theorem (Tietze)

Let  $G = \langle S | R \rangle$  and let  $\langle S' | R' \rangle$  be the result of one of the following transformations.

Then  $\langle S' | R' \rangle \cong G$ .

(T1)  $R' = R \cup \{r\}$ ,  $r \in A(R)^*$

(T2)  $R' = R \setminus \{r\}$ ,  $r \in A(R \setminus \{r\})^*$

(T3)  $S' = S \cup \{x\}$ ,  $R' = R \cup \{x^{-1}w\}$  for any  $w \in A(S)^*$

(T4)  $S' = S \setminus \{x\}$ ,  $R' = R \setminus \{r\}$  when  $x$  occurs only once in  $r$  and does not occur in  $R$ .

□

Suppose that  $\langle S_1 | R_1 \rangle \cong \langle S_2 | R_2 \rangle$ .

Theorem (Tietze): there is a word  $w \in A(\{T_1, \dots, T_4\})$   
which transforms  $\langle S_1 | R_1 \rangle$  to  $\langle S_2 | R_2 \rangle$ .

Proof: If  $\varphi: \langle S_1 | R_1 \rangle \rightarrow \langle S_2 | R_2 \rangle$  is an  
isomorphism, then

$\varphi(s)$  is a word in  $A(S_2)^*$ .

$$\begin{array}{ccc}
 \langle S_2 | R_2 \rangle & \xrightarrow{T_{3s}} & \langle S_1 \cup S_2 | R_2 \cup \underbrace{\{s^{-1}\varphi(s) : s \in S_1\}}_{R_3} \rangle \\
 & & \downarrow T_{1s} \\
 \langle S_1 \cup S_2 | R_1 \cup R_3 \rangle & \xleftarrow{T_{2s}} & \langle S_1 \cup S_2 | R_1 \cup R_2 \cup R_3 \rangle \\
 \downarrow T_{4s} & & \\
 \langle S_1 | R_1 \rangle & & 
 \end{array}$$

□

Again with this "algorithm" we face a problem  
of graph exploration. However there are good  
heuristics to locally "minimize" a presentation:

only look for "local minima".

More useful transformations (human readable):

(T1'): replace  $r$  by its (minimal) cyclicly reduced form.

(T2'): if  $r_1 = abc$ ,  $r_2 = dbf$  ( $a, b, c, d, f \in \mathcal{A}(S)^+$ )

$\Rightarrow b = d'f'$   $\Rightarrow r_1' = ad'f'c$  & remove  $r_2$

(T3'):  $\exists r = axb$ ,  $a, b \in \mathcal{A}(S \setminus \{x\})^+$

$\Rightarrow x = a'b'$   $\Rightarrow$  remove  $x$  from  $S$  & replace occurrences of  $x$  with  $a'b'$ .

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In practice more than just local minima are explored:

- 1) eliminate gens using rels of length 1 or 2
- 2) eliminate <sup>some</sup> generators using (T3') at the cost of increasing the presentation length
- 3) find common intersections of relators to eliminate them using (T2').

