Given $G < \mathrm{Sym}(n)$ split it into

$N \triangleleft G$ and G/N i.e.

$$0 \longrightarrow \underset{\substack{\shortparallel \\ \ker\varphi}}{N} \longrightarrow G \xrightarrow{\varphi} G/N \longrightarrow 1$$

faithfully

$$G \curvearrowright \Omega = \Delta \sqcup \Gamma$$

disjoint union of G-invariant sets

$$\Rightarrow \quad G \xrightarrow{\alpha} \mathrm{Sym}(\Delta)$$
$$G \xrightarrow{\beta} \mathrm{Sym}(\Gamma)$$

action homomorphisms

satisfy $\ker\alpha \cap \ker\beta = \langle 1 \rangle$.

$$\mathrm{im}\,\alpha = A < \mathrm{Sym}(\Delta)$$
$$\mathrm{im}\,\beta = B < \mathrm{Sym}(\Gamma)$$

$$\varphi : G \longrightarrow A \times B$$
$$g \longmapsto (\alpha(g), \beta(g))$$

$\ker\varphi = \langle 1 \rangle$.

$$\begin{tikzcd} G \arrow[dr, hook, "\varphi", "\text{inj.}"'] & & \\ & A \times B \arrow[r, "\pi_2", "\text{epi}"'] \arrow[d, "\pi_1", "\text{epi}"'] & B \\ & A & \end{tikzcd}$$

<u>Defn.</u> We say that G is (isomorphic to) a <u>sub-direct product.</u>

$$
\begin{tikzcd}
 & \ker\alpha \arrow[d, dashed] & & \\
\ker\beta \arrow[r, dashed] & G \arrow[dr, "\varphi"] \arrow[drr, dashed, "\beta"] \arrow[ddr, dashed, "\alpha"'] & & \\
 & & A\times B \arrow[r] \arrow[d] & B \arrow[d] \\
 & & A \arrow[r] & A/\alpha(\ker\beta) \cong B/\beta(\ker\alpha)
\end{tikzcd}
$$

$$\ker\alpha < G \Rightarrow \beta(\ker\alpha) \triangleleft \beta(G) = B$$

$$\alpha(\ker\beta) \triangleleft \alpha(G) = A$$

$$
\begin{tikzcd}
 & G/\langle \ker\alpha, \ker\beta \rangle \arrow[dl, "\alpha"] \arrow[dr, "\beta"] & \\
A/\alpha(\ker\beta) \arrow[rr, "\cong"] & & B/\beta(\ker\alpha)
\end{tikzcd}
$$

$$\left(G/\ker\alpha\right)/\ker\beta \cong G/\langle \ker\alpha, \ker\beta \rangle \cong$$

$$\cong \left(G/\ker\beta\right)/\ker\alpha$$

$$\zeta : A/_{\mathrm{im}\,\alpha(\ker\beta)} \xrightarrow{\cong} B/_{\mathrm{im}\,\beta(\ker\alpha)}$$

$$\begin{tikzcd}
 & a' \arrow[rr, "\zeta"] & & b' & \\
x \in \ker\beta & A \ni a\cdot\alpha(x) \arrow[u, squiggly] & & b\cdot\beta(y) \arrow[u, squiggly] & y \in \ker\alpha \\
 & G \ni g\cdot y\cdot x \arrow[u, squiggly] \arrow[rr, "\beta"] & & \underline{\beta(g)}\cdot\beta(y) \arrow[u, squiggly] &
\end{tikzcd}$$

$x \in \ker\beta$

$y \in \ker\alpha$

thus if $\quad g \in \mathrm{im}\,\varphi(G) \in A\times B$

then $\quad \zeta(\alpha(g)) = \beta(g)$

<u>Definition</u> : <u>Pullback</u> (direct product with amalgamation, external subdirect product)

A, B two groups s.t.

$D \triangleleft A$, $E \triangleleft B$. and $\zeta : A/_D \xrightarrow{\cong} B/_E$.

then $\underline{A \otimes_\zeta B = \{ (a,b) \in A\times B : \zeta(aD) = bE \}}$

Let $\rho : A \to Q$, $\sigma : B \to Q$ be epimorphisms.

$$\begin{tikzcd} A \times_Q B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & A \arrow[d, "\rho"] \\ B \arrow[r, "\sigma"'] & Q \end{tikzcd}$$

let $A \times_Q B$ be the subdirect product, i.e.

$$A \times_Q B = \{ (a,b) \in A \times B : \rho(a) = \sigma(b) \}$$

Suppose that there exist G s.t.

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\rho"] \\ B \arrow[r, "\sigma"'] & Q \end{tikzcd}$$

commutes $(\rho(\alpha(g)) = \sigma(\beta(g)) \ \forall g \in G)$.

Then there exists a unique homomorphism

$\mu : G \to A \times_Q B$ s.t.

$$\begin{tikzcd} G \arrow[dr, "\mu"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\ & A \times_Q B \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & A \arrow[d, "\rho"] \\ & B \arrow[r, "\sigma"'] & Q \end{tikzcd}$$

commutes.

Proof:

Two statements:

1) $\exists\, \mu : G \to A \times_Q B$

$\exists\ \varepsilon : G \longrightarrow A \times B$

$g \longmapsto (\alpha(g), \beta(g))$

$$\begin{tikzcd}
G \arrow[dr] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& A \times B \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & A \arrow[d, "\varrho"] \\
& B \arrow[r, "\sigma"'] & Q
\end{tikzcd}$$

by commutativity: $\forall g\ \varrho(\alpha(g)) = \sigma(\beta(g))$

$\Longrightarrow\ \varepsilon(G) \leq A \times_Q B.$

---

2) let μ be the (co-)restriction of ε.

Suppose that $\mu' : G \to A \times_Q B$ is such map.

$\mu'(g) = (a', b')$. By commutativity (of triangles)

$a' = \pi_1(\mu'(g)) = \alpha(g)$

$b' = \pi_2(\mu'(g)) = \beta(g)$ i.e. $\mu' \equiv \mu$.

Corollary:

Every intransitive perm. group is a sub-direct product of two perm. groups of lower degree.

Imprimitive groups:

- $G \curvearrowright \Omega$ transitively

Defn:

Let $\mathcal{B} = \{B_1, \dots, B_k\}$ $B_i \subset \Omega$; $B_i \cap B_j = \emptyset$ for $i \neq j$
$\bigcup B_i = \Omega$ (i.e. $\mathcal{B}$ is a partition of Ω).

$\mathcal{B}$ is a block system for $G \curvearrowright \Omega$ when
$B_i^g \in \mathcal{B}$ $\forall g \in G$
(i.e. $\mathcal{B}$ is G-invariant).

Ex: trivial block systems: $\mathcal{B}_1 = \{\{i\} : i \in \Omega\}$

$\mathcal{B}_\infty = \{\Omega\}$.

Ex: $G = \langle (1,2,3,4) \rangle$
$\mathcal{B} = \{\{1,3\}, \{2,4\}\}$

Defn: We say that $G \curvearrowright \Omega$ imprimitively iff $G \curvearrowright$ transitively and admits a non-trivial block system.

(Otherwise we say that $G \curvearrowright$ primitively).

Lemma: Let $\mathcal{B} = \{B_1, \dots, B_k\}$ be a block system for $G \curvearrowright \Omega$. The action of G on blocks is transitive.

Proof: transitive = $\forall 1 \leq i,j \leq k \ \exists g \in G$ s.t. $B_i^g = B_j$.

Let $\delta \in B_i$ and $\gamma \in B_j$. since $G \curvearrowright \Omega$ transitively $\Rightarrow \exists g \in G$ s.t. $\delta^g = \gamma$. the same g moves B_i to B_j.

Corollary:

- $|\Omega| = |\mathcal{B}| \cdot |B_1|$
- Block system is determined by a single block.
- If $|G| = p$ (prime); $G \curvearrowright \Omega$ transitively, then $G \curvearrowright$ primitively.

Lemma: Suppose $G \curvearrowright \Omega$ transitively and let $S = \mathrm{stab}_G(x)$ for some $x \in \Omega$.
then there is a bijection between subgroups

$$\{T : S \leq T \leq G\}$$

and

Block systems $\mathcal{B}\{B_1, \dots, B_k\}$ for $G \curvearrowright \Omega$.
namely if $x \in B_1$ then

$$\mathcal{B} \longleftrightarrow \mathrm{stab}_G(B_1).$$

Proof:

Let $S \leq T \leq \mathcal{G}$.

Set $B = x^T$, $\mathcal{B} = B^{\mathcal{G}}$.

Claim: $\mathcal{B}$ is a block system for $\mathcal{G} \curvearrowright \Omega$.

Let $B^g \cap B^h \neq \emptyset$ i.e. $\delta^g = \gamma^h$ for some $\delta, \gamma \in B$.

Since $B = x^{\mathcal{G}}$ $\Rightarrow$ $\delta = x^a$, $\gamma = x^b$ for some $a, b \in T$.

$\Rightarrow$ $x^{a \cdot g} = x^{bh}$ i.e $x^{a \cdot g \cdot h^{-1} \cdot b^{-1}} = x$

$\Rightarrow$ $\underline{a} \cdot g \cdot h^{-1} \cdot \underline{b}^{-1} \in \mathrm{Stab}_{\mathcal{G}}(x) \leq T$

$\Rightarrow$ $gh^{-1} \in T$.

Since T stabilizes B $\quad B^{gh^{-1}} \cap B = B$ i.e.

$$B^g = B^h.$$

$\mathcal{G}$- invariance is obvious by the definition of $\mathcal{B}$.

Let $\mathcal{B}$ be a block system, $x \in B_1 \in \mathcal{B}$.

Any $g \in \mathcal{G}$ which fixes x stabilizes B_1 i.e.

$$\mathrm{Stab}_{\mathcal{G}}(x) \leq \mathrm{Stab}_{\mathcal{G}}(B_1).$$

Let $\delta, \gamma \in B_1$ $\Rightarrow$ $\exists\, g \in \mathcal{G}$ s.t. $\gamma = \delta^g$, which means that $\gamma \in B_1^g \Rightarrow g \in \mathrm{Stab}_{\mathcal{G}}(B_1)$.

$x^{\mathrm{Stab}_{\mathcal{G}}(B_1)} = B_1$.

To finish it's enough to prove that $\mathrm{Stab}_{\mathcal{G}}(x^T) = T$.

If $g \in \mathcal{G}$ s.t. $\forall\, t \in T\ \exists t' \ (x^t)^g = x^{t'}$ then $tgt'^{-1} \in \mathrm{Stab}_{\mathcal{G}}(x^T)$

$\square$

Definition:

A subgroup $S < G$ is called maximal if $S \neq G$ and there is no subgroup $T < G$ s.t. $S \lneq T$.

Corollary:

- A transitive group G is primitive iff a point stabilizer is a maximal subgroup.
- Subgroup $S < G$ is maximal iff $G \curvearrowright S \backslash G$ is primitive.

Finding blocks:

- Let $X \subset \Omega$ be a subset (a seed) we want to be $\subset B_1$.
- Start with $B_X = X$, $\{B_y = \{y\} \text{ for } y \in \Omega \setminus X\}$
- Act via $G = \langle S \rangle$ on each of B_i and merge those which intersect.
- Each B_i is represented by a unique element $\rightarrow$ the representative
- for each $x \in \Omega$ we store the representative of B_i which x belongs to.

ALGORITHM: Union!

Input: C_1 – a subset of Ω

C_2 – a subset of Ω

Output: the union of C_1 and C_2

```
begin
  r1 = representative(C1)
  r2 = representative(C2)
  if r1 ≠ r2
    for x ∈ Ω
      if representative(x) = r2
        set_representative!(x, r2)
      end
  end end
  return C1
end
```

ALGORITHM: Block_system

INPUT:
- S – a generating set for $G = \langle S \rangle$
- Ω – a set with G-action
- $\tilde{B}_1$ – a subset of Ω

OUTPUT: $\mathcal{B}$ – the finest Block system for $G \circlearrowright \Omega$ s.t. $\tilde{B}_1 \subset B_1$

```
begin
  r = [ ]
  queue = [ ]   # a queue of points that have changed
  for x ∈ Ω         their blocks.
    if x ∈ B̃1
      r[x] = first(B̃1)
    else push x to queue
      r[x] = x
    end
end
```

```
for x ∈ q
    δ = representative(x)
    for s ∈ S
        α = representative(x^s)
        β = representative(δ^s)
        if α ≠ β
            Union(α, β)
            push β to queue
        end
    end
end
return r
end
```

Exercise: Rewrite this algorithm using Union-find tree-like data structure.

Theorem:

The algorithm converges.

Proof: Since we're taking only unions the returned partition contains $\tilde{B}_1$ in one of its blocks.

Since every Union! moves up in the lattice of all partitions of Ω and $\{\Omega\}$ - the trivial block system is the maximal element the algorithm has to stop.

unique

We need to prove that the returned partition is in fact a block system i.e. it is <u>G-invariant.</u>

$\forall\ x, y \in B_i \in \mathcal{B} \quad \forall s \in S$

x^s and y^s are in the same cell B_i^s.

Note: it's enough to enforce this for x = representative(B_i), i.e. $y \to x$.

Observe that the queue collects all points for which <u>the condition</u> must be enforced.

If we've added all points whose representative is β we would be ok.

let $\omega \to \beta$. If we reassign $\omega \to \alpha$ this is in call to <u>union!</u>, so at the same time as $\beta \to \alpha$.

therefore enforcing the condition for $\beta \to \alpha$ will automatically do so for $\omega \to \alpha$

<u>Lemma:</u> Let $1 \in B_1 \in \mathcal{B}$ ← block system for $G \curvearrowright \Omega$.

$\Rightarrow B_1$ is union of orbits of $\mathrm{Stab}_G(1)$.

<u>Proof:</u> Let $g \in \mathrm{Stab}_G(1)$ and $\alpha, \beta \in \Omega$ s.t. $\alpha^g = \beta$.

If $\{1, \alpha\} \subset B_1 \Rightarrow \{1, \beta\} \subset B_1^g \Rightarrow B_1 \cap B_1^g \neq \emptyset \Leftrightarrow B_1 = B_1^g$

$\Rightarrow \beta \in B_1$ $\square$

Corollary: It is enough to start with

$$B = \{1\} \cup \alpha^{\mathrm{Stab}_G(1)}$$

since this will be definitely contained in any block that contains $\{1, \alpha\}$.

Practical tip: It's enough to find a few random elements from $\mathrm{Stab}_G(1)$ to compute the orbit of α.

→ e.g. Schreier generators from the transversal.

Recall: Every intransitive group is a sub-direct product of its transitive parts.

Is there a universal description for imprimitive groups?

Defn: Let G, H be groups. Let $\varphi : H \to \mathrm{Aut}(G)$ be a homomorphism. Then

$$G \rtimes_\varphi H = \langle (g,h) \in G \times H, \cdot_\varphi, (1,1) \rangle$$

is a group with

$$(g_1, h_1) \cdot_\varphi (g_2, h_2) = (g_1 \cdot \varphi(h_2)(g_2), h_1 \cdot h_2)$$

$$(g_1, h_1) \cdot_\varphi (g_1, h_1)^{-1} = (1, 1)$$

$$(\varphi(h_1)(g_1^{-1}), h_1^{-1})$$

Defn: Wreath product of G and

permutation group $\rightarrow H < Sym(n)$ is

$$G \wr H := G^n \rtimes H$$

the natural action of H on coordinates of G^n.

$$((g_1, \dots, g_n), h) \cdot ((a_1, \dots, a_n), b) =$$

$$= ((g_1, \dots, g_n) \cdot (a_1, \dots, a_n)^b, h \cdot b) =$$

$$= ((g_1 \cdot a_{b^{-1}(1)}, g_2 \cdot a_{b^{-1}(2)}, \dots, g_n \cdot a_{b^{-1}(n)}), h \cdot b).$$

Suppose that $G \curvearrowright \Omega$. $\Rightarrow G \wr H \curvearrowright \bigsqcup_n \Omega$

i-th copy acts on i-th copy of Ω

H permutes Gs $\rightarrow$ H permutes copies of Ω.

$$\boxed{\text{the imprimitive action of } G \wr H}$$

Theorem: Let G be a transitive, imprimitive group. Let $\mathcal{B}$ be a non-trivial block system for G.

Let $1 \in B \in \mathcal{B}$ and let $T = \mathrm{Stab}_G(B) < G$.

Let $\psi : G \longrightarrow \mathrm{Sym}(\mathcal{B})$ (action homomorphism)

$\varphi : T \longrightarrow \mathrm{Sym}(B)$ (action homomorphism)

Then $G \xhookrightarrow{\mu} \varphi(T) \wr \psi(G)$ as follows:

$$\left\{\begin{array}{l}\text{Let } r_1, \dots, r_n \text{ be representatives for } T\backslash G. \\ \text{Let } g \in G. \text{ then } g \text{ "permutes" the} \\ \quad \text{cosets i.e. } r_i \cdot g \text{ belongs to } \psi(g)(i)\text{-th} \\ \quad \text{coset.} \\ \quad \Rightarrow \quad r_i \cdot g \cdot r_{\psi(g)(i)}^{-1} =: \tilde{g}_i \text{ stabilizes} \\ \quad \text{i-th coset.} \\ \quad \Rightarrow \tilde{g}_i \in T\end{array}\right.$$

we define

$$\mu(g) = \left(\left(\varphi(\tilde{g}_1), \dots, \varphi(\tilde{g}_k)\right); \psi(g) \right)$$

that's almost correct...

$$\varphi(\tilde{g}_i) \rightsquigarrow \varphi\left(\widetilde{g_{\psi(g)^{-1}(i)}}\right)$$

Exercise: check that μ is actually a homomorphism.

Classification of primitive groups

Lemma:

If $G \curvearrowright \Omega$ transitively an $N \triangleleft G$ then orbits of $N \curvearrowright \Omega$ form a block system for $G \curvearrowright \Omega$.

Proof: Δ - N-orbit, $g \in G$.

We will show that Δ^g is N- orbit.

If $\delta, \gamma \in \Delta \Rightarrow \delta^g, \gamma^g \in \Delta^g$.

Let $n \in N$ s.t. $\delta^n = \gamma$ then

$(\delta^g)^{\underbrace{g^{-1}ng}_{m \in N}} = \delta^{ng} = \gamma^g \Rightarrow \gamma^g$ and δ^g are in the same N orbit.

If Δ^g is not the whole N-orbit

then $(\Delta^g)^{g^{-1}}$ is a proper subset of Δ $\lightning$

$\square$.

Corollary: If G is primitive then N acts transitively.

Defn:

$N \triangleleft G$ is called <u>minimally normal</u> if the only normal in G proper subgroup of N is $\{1\}$.

$(M \triangleleft G$ & $M \lneq N \Rightarrow M = \{1\})$

Lemma:

Minimally normal subgroups are of the form

$$N = \bigoplus_k T$$

where T is a simple group.

Definition: The socle of G is the subgroup generated by minimally normal subgroups:

$$\mathrm{Soc}(G) = \langle N \mid N \trianglelefteq G \text{ - minimally} \rangle.$$

Lemma:

$$\mathrm{Soc}(G) = \bigoplus_i N_i \qquad N_i \text{ - minimally normal}$$

Proof:

If $H = \langle N, M \rangle$ (subgroups) and $N \cap M = \langle 1 \rangle \Rightarrow$

$$H \cong N \oplus M$$

Let M - maximal subgroup of $\mathrm{Soc}(G)$ which is a product of minimally normal subgroups.

If $M \neq \mathrm{Soc}(G) \Rightarrow \exists\, N \trianglelefteq G$ s.t. $N \not\leq M$.
minimally normal.

then $N \cap M \trianglelefteq G$

by minimality of N : $N \cap M = \langle 1 \rangle$

$\Rightarrow \langle N, M \rangle = N \times M$

$\lightning$. $\square$

Lemma:

Let $G \curvearrowright \Omega$ primitively.

$\Rightarrow$ $\mathrm{Soc}(G)$ is minimally normal or

$\mathrm{Soc}(G) = N \times M$, $N \cong M$ minimally normal and non-abelian.

Types of Socles:

- $soc(G) \cong \bigoplus_m T$ ← homogeneous of type T

↳ T - abelian i.e. $T \cong C_p$ - cyclic of order p.

⇒ primitive $G \cong soc(G) \rtimes Stab_G(1)$

Proof:

$Soc(G)$ - abelian, minimally normal ⇒

$Soc(G) \cong C_p^m \cong \mathbb{F}_p^m$. By transitivity & faithfullness

p - prime

$|\Omega| = p^m$.

Let $S = Stab_G(1)$ ← by primitiveness S is a maximal subgroup, but

$soc(G) \not\leq S$ ($soc(G) \curvearrowright \Omega$ transitively !)

⇒ $G = soc(G) S$. Since $soc(G)$ is abelian

$S \cap soc(G) \triangleleft soc(G)$. Since we also have

$S \cap soc(G) \triangleleft S$ ⇒ $S \cap soc(G) = \langle 1 \rangle$

⇒ $G \cong soc(G) \rtimes S$.

□

Note: the action of G on Ω is through an affine map where each element of S acts through a matrix representation

$$S \longrightarrow GL(m, \mathbb{F}_p)$$

and $soc(G)$ corresponds to translations.

- $Soc(G)$ is non-abelian.

  ↳ If $Z(soc(G)) = \langle 1 \rangle \Rightarrow G \leq Aut(soc(G))$.

  Proof: If $G \curvearrowright soc(G)$ by conjugation, then $C_G(soc(G))$ is the kernel of the action.

  $Z(soc(G)) = \{ z \in soc(G) \mid \forall g \in soc(G) \; gz = zg \}$

  is a normal subgroup of $C_G(soc(G))$.

  Any minimally normal subgroup of $C_G(soc(G))$ would be inside $Z(soc(G))$ which is trivial

  $\Rightarrow$ the kernel of the action of G on $soc(G)$ is trivial $\Rightarrow$

  $$G \hookrightarrow Aut(soc(G)).$$

Lemma: for T - simple $Aut(T^m) = Aut(T) \wr Sym(m)$.

$\square$

Product action of $G \wr H$:

$G \leq Sym(\Omega)$

$H \leq Sym(\Delta)$ $\Rightarrow G \wr H \curvearrowright \Omega^\Delta \cong \Omega^{|\Delta|}$

$|\Delta|$-tuples of elts from Ω

$d = |\Delta|$

$G^d \curvearrowright \Omega^d$ "dimension-wise"

$H \curvearrowright \Omega^d$ "permuting the dimensions"

$\boxed{\text{the product action of } G \wr H}$.

Let $\mathcal{D} < T^m$ be the image of

$$T \hookrightarrow T^m \qquad g \mapsto (g, \dots, g).$$ (diagonal embedding).

$$T^m \curvearrowright \underbrace{\mathcal{D}\backslash T^m}_{\Omega}$$ (action homomorphism into $\mathrm{Sym}(|\mathcal{D}\backslash T^m|)$ of degree $n = |T|^{m-1}$.)

$$\mathcal{N} = \mathcal{N}_{\mathrm{Sym}(n)}(T^m) \curvearrowright T^m \text{ by conjugation}$$

$$\mathcal{N} \triangleleft \mathrm{Aut}(T^m)$$

However we don't necessarily have $\mathcal{G} < \mathcal{N}$.
In the case this happens we say that $\mathcal{G}$ is of <u>diagonal type</u>.

<u>Lemma:</u>
$\mathcal{G}$ of diagonal type is primitive iff $m = 2$ or $\mathcal{G} \curvearrowright T^m$ is primitive.

Theorem: (Scott - O'Nan theorem).

$G \curvearrowright \Omega$ primitively, faithfully with $|\Omega| = n$.
Let $H = soc(G)$ and assume that $H = T^m$ is of type T. Then one of the points below describes the action:

1) T is abelian of order p, $n = p^m$,
$G \cong H \rtimes Stab_G(x) \quad (x \in \Omega)$
$G \curvearrowright \Omega$ through an affine action.

2) $m = 1$, $H \trianglelefteq G \leq Aut(H)$ "G is almost simple".

3) $m \geq 2$, $n = |T|^{m-1}$, $G \leq Aut(T) \wr Sym(m)$
and either
3a) $m = 2$ $G \curvearrowright \{T_1, T_2\}$ intransitively
3b) $m \geq 2$ $G \curvearrowright \{T_1, \dots, T_m\}$ primitively
the action of G on Ω is of the diagonal type.

4) $m = rs$, $s > 1$ then
$G \leq A \wr B$ where $A \wr B$ acts through the product action.
Therefore $B < Sym(d)$, $|\Omega| = n = d^s$ and B is transitive. A is primitive
4a) of type 3a with $soc(A) = T^2$ (i.e. $r = 2$)
4b) of type 3b with $soc(A) = T^r$
4c) of type 2 (i.e. $r = 1$, $s = m$).

5) "Twisted wreath type". H is freely and $n = |T^m|$.

$\mathrm{Stab}_G(x)$ is a transitive subgroup of $\mathrm{Sym}(m)$.

(note: this type occurs only for groups of order 60^6).