Small hyperbolic groups with property (T)

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Outline

Definitions

Generalized triangle groups

An example

A bit about proof

Definitions

Definition (Cayley graph)

Let $G = \langle S | \mathcal{R} \rangle$ be a (finitely presented) group. Cayley graph $\mathrm{Cay}(G, S)$ is a graph (V, E), where

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- Global assumptions: groups are finitely presented, generated by a symmetric set which doesn't contain the identity.

Hyperbolic groups

Definition (Hyperbolic group)

- A group (G, S) is (word) hyperbolic when there exists $\delta > 0$ such that $\mathrm{Cay}(G, S)$ is a δ-hyperbolic metric space.
- A graph is δ-hyperbolic if for every geodesic triangle δ-neighbourhood of two edges contains also the third one.

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Are hyperbolic groups *residually finite*?

Property (T) – meaningless gist

- Property (T) is an analytic property defined in terms of unitary actions;
- There is a constant $\kappa(G, S) \geqslant 0$ (think: universal spectral gap of group Laplacian for any $*$-representation) which is indicator of the property;

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How can we find a hyperbolic group which has property (T)?

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Definition

Take $G = C_k * C_k * C_k = \langle a,b,c | 1 = a^k = b^k = c^k \rangle$ and specify three groups:

- $L_{a,b} \lhd \langle a,b \rangle < G$,
- $L_{b,c} \lhd \langle b,c \rangle < G$,
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Then

$$G(L_{a,b}, L_{b,c}, L_{a,c}) = G / \langle L_{a,b}, L_{b,c}, L_{a,c} \rangle$$

is **generalized** k**-fold triangle group**.

An example

Theorem (Lubotzky-Manning-Wilton, 2019)

There exist generalized 18*-fold triangle groups which are hyperbolic and have property (T).*

Question: *What is the lowest* $k \geqslant 3$ *such that such examples exist?*

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I.e. for $G = C_5 * C_5 * C_5 = \langle a, b, c | 1 = a^5 = b^5 = c^5 \rangle$ and

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$G(L_{a,b}, L_{b,c}, L_{a,c})$ is hyperbolic and has property (T).

A bit about proof: hyperbolicity

- The group comes as a group of a poset of groups over a triangle (trivial facet group, cyclic groups on edges, finite groups on edges).
- To every poset of groups there exists a canonical construction of a space (a simplicial complex) the poset group acts property on (Haefliger-Brideson).
- Let

$$\Gamma_{a,b} = \Gamma_{\langle a,b\rangle / L_{a,b}}(\{a, \dots a^4, b, \dots b^4\})$$

  denote the (bipartite) coset graph. The geometry of the space is determined by links of its vertices, which are precisely graphs $\Gamma_{a,b}, \Gamma_{a,c}, \Gamma_{b,c}$.
- their girths are $\gamma_{a,b} = 4$, $\gamma_{a,c} = 6$ and $\gamma_{b,c} = 14$, and

$$\frac{2}{\gamma_{a,b}} + \frac{2}{\gamma_{a,c}} + \frac{2}{\gamma_{b,c}} = \frac{41}{42} < 1.$$

A bit about proof: Property (T)

Corollary (to a theorem of Ershov & Jaikin-Zapirain, 2010)

Let $G = \langle A_a, A_b, A_c \rangle$ be generated by three (finite) subgroups such that $X_{i,j} = \langle A_i, A_j \rangle$ is finite for each $\mathbf{i}, \boldsymbol{j}$. If

$$\varepsilon_{X_{a,b}}^2 + \varepsilon_{X_{a,c}}^2 + \varepsilon_{X_{b,c}}^2 + 2\varepsilon_{X_{a,b}}\varepsilon_{X_{a,c}}\varepsilon_{X_{b,c}} < 1$$

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Theorem (Dymara-Januszkiewicz, 2002; Oppenheim, 2017)

Constant $\varepsilon_{X_{i,j}}$ (so called angle between subgroups A_i, A_j) is equal the spectral gap of the (combinatorial) Laplacian on the link graph $\Gamma_{i,j}$.

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- $\varepsilon_{C_5 \oplus C_5} = 0$
- $\varepsilon_{H_3(\mathbb{F}_p)} = \frac{1}{\sqrt{p}}$ (Ershov & Jaikin-Zapirain)

A bit about proof: Property (T)

- $\Gamma_{i,j} = Cay(X_{i,j}, (A_i \cup A_j) \setminus \{1\})$
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- $PSL_2(\mathbb{F}_{109})$ is a finite group, so has a finite number of irreducible representations (each of degree $\leqslant 110$).
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- the largest spectral gap is afforded by the principal representation associated to character $\nu_5 : \mathbb{F}_{109}^{\times} \to \mathbb{C}$, defined by $\nu_5(\alpha) = \zeta_{54}^5$ (where $\alpha = 6$ was chosen as the generator of $\mathbb{F}_{109}^{\times}$). It fits

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which results in

$$\varepsilon_{X_{a,b}}^2 + \varepsilon_{X_{a,c}}^2 + \varepsilon_{X_{b,c}}^2 + 2\varepsilon_{X_{a,b}}\varepsilon_{X_{a,c}}\varepsilon_{X_{b,c}} \in 0.779751342877069635 9 \pm 4.45 \cdot 10^{-20} < 1$$

Dziękuję!