On property (T) for $\mathrm{Aut}(F_n)$ and $\mathrm{SL}(n, \mathbb{Z})$

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Outline

Group Laplacians & Property (T)

Positivity & SDP

The procedure

How to square the Laplacian?

Results

Notation / Convention

- $G = \langle S \mid R \rangle$ is a finitely presented group generated by a fixed **symmetric generating set** (i.e. $S^{-1} = S$).

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- $G = \mathrm{SAut}(F_n)$ or $G = \mathrm{SL}(n, \mathbb{Z})$ in this talk.

Group Laplacians

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- spectrum of Δ is real and non-negative;
- the second eigenvalue λ_1 is called the **spectral gap**

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots$$

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- We define

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over all orthogonal representations π of G. We say that G has the **Kazhdan's property (T)** if and only if there exists a (finite) generating set S such that $\kappa(G, S) > 0$.

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Random group elements in finite groups:

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*Let K be a finite group generated by $k \leq n$ elements. If $\mathrm{SAut}(F_n)$ has property (T) with constant $\kappa = \kappa(\mathrm{SAut}(F_n), \{\mathrm{transvections}\}) > 0$, then **PRA walk** on $\Gamma_n = \Gamma_n(K)$ has fast mixing rate, i.e.*

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$$\left\|Q^t_{(g)} - U\right\|_{\mathrm{tv}} \leqslant \varepsilon \qquad \textit{for} \qquad t \geqslant \frac{16}{\kappa^2} \log \frac{|\Gamma_n|}{\varepsilon} \sim O\left(\left(\frac{n}{\kappa}\right)^2 \log \frac{|K|}{\varepsilon}\right)$$

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Note (Motivation)

We do observe fast mixing rate in practice for large n.

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- Connection to eigenvalues of Δ: $\kappa(G, S)$ can be estimated as

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How to prove that $\Delta^2 - \lambda\Delta \geqslant 0$?

How to prove that a polynomial $f \geqslant 0$?

Positivity & SDP

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- find a sum of squares decomposition of $\boldsymbol{f} \in \mathbb{R}[x_1, \ldots, x_n]$, i.e.

$$f = \sum_i f_i^2, \quad f_i \in \mathbb{R}[x_1, \ldots, x_n].$$

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$x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \geqslant \mathbf{0}$ but not SOS.

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$$p \geqslant 0 \iff \exists q \colon q^2 p \in \Sigma^2 \mathbb{R}[x_1, \dots x_n]$$

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$\left(x^2 + y^2 + 1\right)\left(x^4y^2 + x^2y^4 - 3x^2y^2 + 1\right)$ is a sum of squares!

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Proof.

If P is positive definite, then $P = Q^T Q$ and

$$f = \vec{\mathbf{x}}^T P \vec{\mathbf{x}} = \vec{\mathbf{x}}^T Q^T \cdot Q\vec{\mathbf{x}} = (Q\vec{\mathbf{x}})^T \cdot (Q\vec{\mathbf{x}}) .$$

□

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Mathematical Programming

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Semi-definite programming

- optimise linear functional
- on a polytope intersected with the cone of SPD matrices (spectrahedron)
- weak duality, non-unique solutions
- even feasibility is a hard problem!

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tries to maximise λ as long as $(2x^2 + 4x + 1) - \lambda \geqslant 0$.

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For any $*$-invariant element $\xi \in \mathbb{R}[G]$

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Let $I[G]$ denote the augmentation ideal of $\mathbb{R}[G]$. Δ is an interior point of $\Sigma^2 I[G] = I[G] \cap \Sigma^2\mathbb{R}[G]$

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Let $I[G]$ denote the augmentation ideal of $\mathbb{R}[G]$. Δ is an interior point of $\Sigma^2 I[G] = I[G] \cap \Sigma^2\mathbb{R}[G]$, i.e. for a $*$-invariant element $\xi \in I[G]$

$$\xi \geqslant 0 \iff \xi + \varepsilon\Delta \in \Sigma^2 I[G]$$

for all $\varepsilon > 0$.

Example

If we can show that $\Delta^2 - \lambda\Delta + \varepsilon_0\Delta = \sum \xi_i^* \xi_i$ for a single fixed ε_0, then

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Corollary (Ozawa, 2015)

The following conditions are equivalent:

- G *has property (T)*
- *there exists* $\lambda > 0$, *and* $\xi_1, \ldots \xi_n \in I[G]$ *such that*

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4. Compute $\sqrt{P} = Q = [\overrightarrow{q_e}, \ldots, \overrightarrow{q_{g_n}}]$

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2. Set $\vec{\mathbf{x}} = (e, g_1, g_2, \ldots, g_n), \quad g_i \in B_d(e, S)$ $(d = 2, 3, \ldots)$;
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$$\begin{aligned} \text{maximize:} \quad & \lambda \\ \text{subject to:} \quad & P \succcurlyeq 0, \quad P \in \mathbb{M}_{\vec{\mathbf{x}}} \\ & \lambda \geqslant 0 \\ & (\Delta^2 - \lambda\Delta)_t = \sum_{g^{-1}h = t} P_{g,h}, \quad \text{for all } t \in B_{2d}(e, S) \end{aligned}$$

4. Compute $\sqrt{P} = Q = [\overrightarrow{q_e}, \ldots, \overrightarrow{q_{g_n}}]$
5. Finally: $\xi_g = \langle \vec{\mathbf{x}}, \overrightarrow{q_g} \rangle$ and $\Delta^2 - \lambda\Delta = \sum_{g \in \vec{\mathbf{x}}} \xi_g{}^* \xi_g$.

Can we certify that the numerical result is sound?

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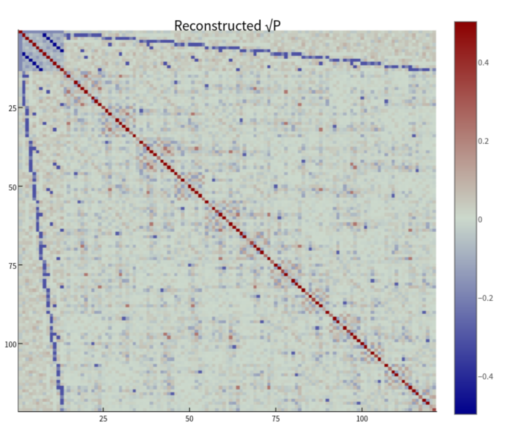
Yes we can!

Example: $SL(3,\mathbb{Z})$



$\sqrt{P} = Q \in \mathbb{M}_{\vec{\mathbf{x}}}$, where $\vec{\mathbf{x}} = B_2(e, E(3))$, i.e. rows and columns are indexed by elements in $(\mathrm{SL}(3,\mathbb{Z}), E(3))$ of word length $\leqslant 2$. In this case

$$\Delta^2 - 0.28\Delta = \sum_{i=1}^{121} \xi_i^* \xi_i + r, \qquad \|r\|_1 \in [3.8508, 3.8511] \cdot 10^{-7}$$

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- Heureka! we found that

$$\Delta_5{}^2 - 1.2999\,\Delta_5 \in \Sigma^2 I\,\mathrm{SAut}(F_5),$$

i.e. $\mathrm{SAut}(F_5)$ has property (T)!

Squaring the Laplacian

Setting the scene

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- each generating set admits action of A_n as the subgroup of the "Weyl group".
- the alternatig group A_n acts on the set E_n of edges of $(n-1)$-dimensional simplex (action on tuples)
- there is an A_n-equivariant function $l_n \colon S_n \to E_n$, which assignes (doubly-indexed) generators to edges of the standard $(n-1)$ simplex.

- for each edge $e \in E_n$ let Δ_e denote the **edge Laplacian**, i.e.

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Moreover, for $\mathbf{m} \geqslant \mathbf{n} \geqslant 3$ we have

$$\sum_{\sigma \in A_m} \sigma(\Delta_n) = \binom{n}{2} \cdot (m-2)! \Delta_m.$$

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- Intuition: Op_n will dominate Adj_n for $n \geqslant 6$.

Reducing to the single case: $\mathrm{SL}(3,\mathbb{Z})$

Lemma

For $m \geqslant n \geqslant 3$ *we have*

$$\sum_{\sigma \in A_m} \sigma(\mathrm{Adj}_n) = n(n-1)(n-2)\frac{(m-3)!}{2}\,\mathrm{Adj}_m\,.$$

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$$\mathrm{Adj}_3 - 0.157\Delta_3 \in \Sigma^2 I[\mathrm{SL}(3,\mathbb{Z})]$$

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therefore for all $m \geq 3$ and some $C_m, \alpha_m > 0$

$$\sum_{\sigma \in A_m} \sigma\left(\mathrm{Adj}_3 - 0.157\Delta_3\right) = C\left(\mathrm{Adj}_m - \alpha_m\Delta_m\right) \in \Sigma^2 I[\mathrm{SL}(m,\mathbb{Z})].$$

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Corollary

$\mathrm{SAut}(F_m)$ *has property (T) for all* $m \geqslant 7$.

(but we can do $n = 6$ as well!)

Kazhdan constants: $\mathrm{SL}(n, \mathbb{Z})$

Żuk, 1999 $\kappa(\mathrm{SL}_n(Z), S_n) \leqslant \sqrt{\frac{2}{n}}$

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These give e.g. for $N = 7$

$$0.00102 \leqslant \qquad \leqslant \kappa(\mathrm{SL}_{n(\mathbb{Z})}, S_n) \leqslant 0.53452.$$

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Results of KKN:

- $0.00983 \leqslant \kappa(\mathrm{SAut}(F_6), S_6)$ (by different method),
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and in general for $N \geqslant 9$:

$$\sqrt{\frac{1.316(n-2)}{6(n^2-n)}} \leqslant \kappa(\mathrm{SAut}(F_n), S_n).$$

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(*The Method of Mechanical Theorems*, Archimedes)