Certifying numerical estimates of spectral gaps

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Outline

Graph Laplacians

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What is a graph Laplacian?

Definition

If Σ is a simplicial complex then

$$\Delta = \partial_1 \circ \delta_1$$

is its Laplacian.

$$\begin{array}{ccc} C^0(\Sigma, \mathbb{F}) & \xrightarrow{\delta_1} & C^1(\Sigma, \mathbb{F}) \\ & & \downarrow \\ C_0(\Sigma, \mathbb{F}) & \xleftarrow{\partial_1} & C_1(\Sigma, \mathbb{F}) \end{array}$$

Alternatively: for $G = \Sigma^{(1)}$

$$\Delta = \deg(G) - \mathrm{adj}(G).$$

$$
\begin{pmatrix}
3 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\
-1 & 3 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 3 & -1 & 0 & 0 & -1 & 0 \\
0 & -1 & -1 & 3 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 3 & -1 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & 3 & 0 & -1 \\
0 & 0 & -1 & 0 & -1 & 0 & 3 & -1 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & 3
\end{pmatrix}
$$

Properties of $\Delta(G)$

1. $\Delta = \partial\delta = \partial\partial^*$ is symmetric, non-negative;
2. eigenvalues of Δ are real and non-negative;
3. eigenvectors of Δ are "smoothly varying functions on G";
4. $\dim\ker\Delta = \dim H_0$;
5. if G is connected then $\lambda_1 > 0$ is called the **spectral gap**

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n.$$

6. (normalised) Δ is a (uniform) random walk operator on G;
7. properties of random walk on G are strongly related to eigenvalues: after

$$s \geq \frac{1}{\lambda_1 \log \frac{\max \sqrt{deg(v)}}{\varepsilon \min \sqrt{deg(x)}}}$$

steps random walk will be ε-close to uniform distribution.

Group Laplacians

Cayley Graphs

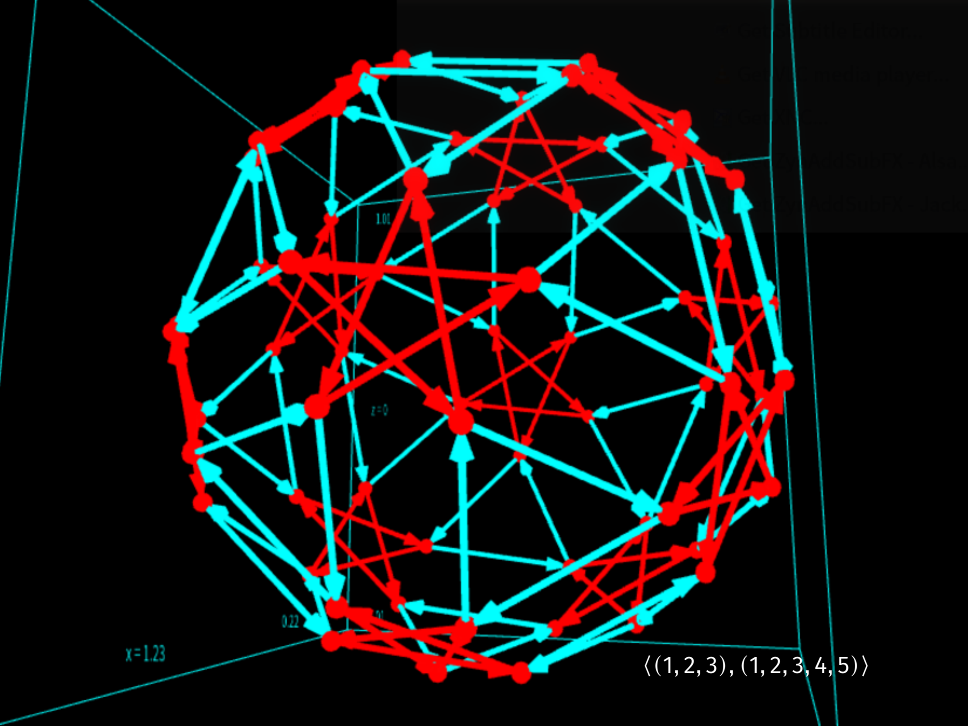
Definition

Let $G = \langle \mathcal{S} | \mathcal{R} \rangle$ be a group. Cayley graph $\mathsf{Cay}(G, \mathcal{S})$ is a graph (V, E), where

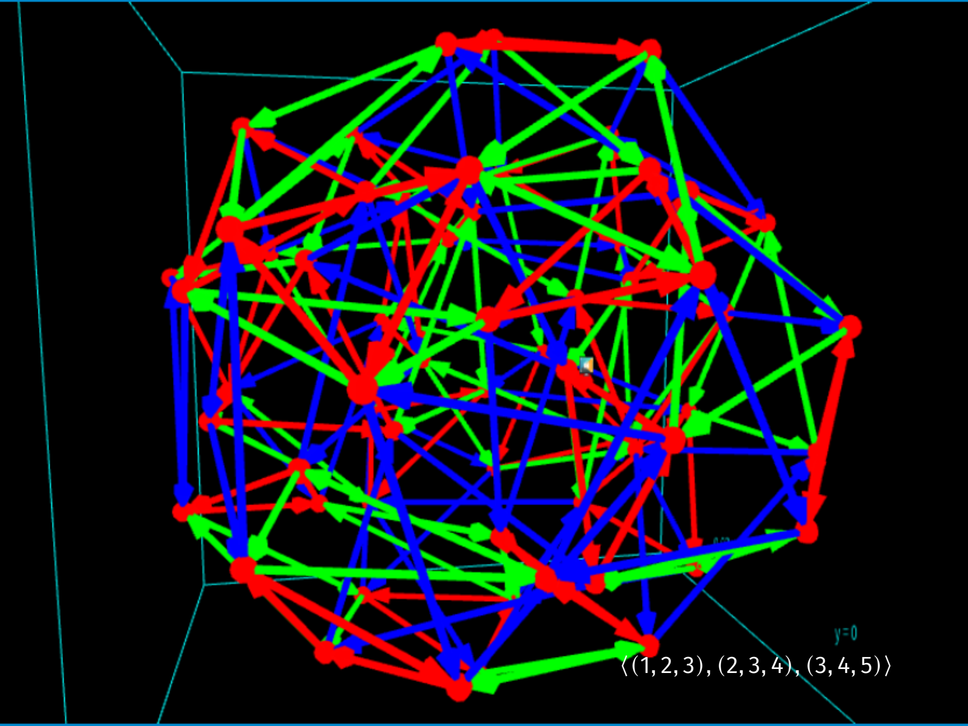
$$V = G \quad \text{and}$$
$$(g, h) \in E \iff gh^{-1} \in \mathcal{S}.$$

We usually assume $\mathcal{S} = \mathcal{S}^{-1}$.

$\mathsf{Cay}(G, \mathcal{S})$ are very symmetric: links of all vertices are isomorphic.



$\langle (1,2,3), (1,2,3,4,5) \rangle$



$\langle (1,2,3), (2,3,4), (3,4,5) \rangle$

Definition

$\Delta(G, \mathcal{S}) = \Delta(\mathrm{Cay}(G, \mathcal{S}))$

Note

$\Delta\colon \mathbb{R}^n \to \mathbb{R}^n$, but vertices are indexed by elements of G:

$$\Delta\colon \mathbb{R}[G] \to \mathbb{R}[G]$$
$$\Delta = |\mathcal{S}|\,\mathrm{Id} - \sum_{g\in\mathcal{S}} g = \frac{1}{2}\sum_{g\in\mathcal{S}} (1-g)^*(1-g)$$

Also: for fixed G and $\mathcal{S}$:

$$\Delta^2 - \lambda\Delta \geq 0 \iff \lambda \in [0, \lambda_1).$$

WHY? **Infinite groups**, e.g. $\mathrm{SL}(n, \mathbb{Z})$'s.

Motivation

- Random group elements (estimating stabilising times)
- Expanders
- Connection to Kazhdan's property (T): constant $\kappa(G, \mathcal{S})$ is estimated:

$$\sqrt{\frac{2\lambda_1(G, \mathcal{S})}{|\mathcal{S}|}} \leq \kappa(G, \mathcal{S}).$$

Positivity

How to prove that a polynomial $f \in \mathbb{R}[x]$ is positive?

Theorem

A the set of points satisfying

$$f_i = 0 \quad \textit{for } i = 1, \ldots, n$$
$$g_j \geq 0 \quad \textit{for } j = 1, \ldots, m$$

is empty if and only if $-1 \in I(f_i) + C(g_j)$ *(positive cone of* g_j*'s), i.e.*

$$-1 = \sum a_i f_i + \sum_{\mu \in \{0,1\}^m} s_\mu g_1^{\mu_1} \cdots g_m^{\mu_m}$$

where $s_\mu \in \Sigma^2$ *(is a sum of squares).*

Easy version: ($i = 0, j = 1$):

a polynomial $g \geq 0$ ***iff it is a sum of squares of rational functions***.

SOS decomposition

Problem

How to find such sum of squares (SOS) decomposition?

By evaluating

$$f(x,y) = (1,x,y)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

we can obtain any polynomial f of degree 2 in x and y.

For f to be a SOS we just need $(a_{ij}) = A \succcurlyeq 0$ to be **semi-positive definite**.

SDP optimisation

Linear programming:

- optimise linear functional
- on the set constrained by hyperplanes (polytope)
- strong duality, unique solution

Semi-definite programming

- optimise linear functional
- on a polytope intersected with the cone of SDP matrices (spectrahedron)
- weak duality, non-unique solutions

SDP problem formulation

Example (SDP problem)

$$\begin{aligned} \text{maximise:} \quad & y \\ \text{subject to:} \quad & c - y = 1 \\ & b_1 + b_2 = -4 \\ & a = 2 \\ & \begin{bmatrix} c & b_1 & 0 \\ b_2 & a & 0 \\ 0 & 0 & -y \end{bmatrix} \succcurlyeq 0 \end{aligned}$$

tries to maximise y as long as $2x^2 - 4x + 1 - y \geq 0$.

Theorem (Helton)
$f \in \mathbb{R}[G]$ *fulfills*

$$f(A_1, \ldots, A_n) \geq 0$$

for all $A_i \in \mathrm{Sym}_s(\mathbb{R})$ *(and all* $s \geq 1$*) if and only if* $f \in \Sigma^2\mathbb{R}[G]$, *i.e.*

$$f = \sum \xi_j{}^* \xi_j, \text{ for some } \xi_i \in \mathbb{R}[G].$$

Note

- $\Delta = \sum_{g \in S}(1-g)^*(1-g) \in \Sigma^2\mathbb{R}[G]$.
- $f(A_1, \ldots, A_n) \geq 0$ for all $A_i \in \mathrm{Sym}_s(\mathbb{R})$ is equivalent to

$$\pi(f) \geq 0 \quad \text{for all } C^*\text{-algebra representations of } G.$$

Action Plan

1. Pick $G = \langle \mathcal{S} | \mathcal{R} \rangle$;
2. Set $E = [e, g_1, g_2, \ldots, g_n]$ for $g_i \in B_2(e, \mathcal{S})$;
3. Solve the problem:

$$\begin{aligned} \text{maximize:} \quad & \lambda \\ \text{subject to:} \quad & P \succcurlyeq 0, \quad P \in \mathbb{M}_E(\mathbb{R}) \\ & \Delta^2 - \lambda\Delta = EPE^T \\ & \lambda \geq 0 \end{aligned}$$

4. Compute $\sqrt{P} = Q = [q_1, \ldots, q_n]$
5. Finally: $\xi_j = \langle E, q_j \rangle$ and $\Delta^2 - \lambda\Delta = \sum {\xi_j}^* \xi_j$.

How do we certify that the result is sound?

Using non-commutative real algebraic geometry.

Proposition (Ozawa)

For a $*$-invariant $\xi \in I[G] \subset C^*G$ the following are equivalent:

- ξ is non-negative
- $\xi + \varepsilon\Delta \in \Sigma^2 I[G]$ for all $\varepsilon > 0$.

Lemma (Netzer&Thom)

Let $r \in I[G] \subset \mathbb{R}[G]$ *such that* $\operatorname{supp}(r) \subset B_d(e)$. *Then*

$$r + 2^{d-1}\|r\|_1 \cdot \Delta \in \Sigma^2\mathbb{R}[G].$$

Action Plan 2

1. Pick $G = \langle \mathcal{S} | \mathcal{R} \rangle$;
2. Set $E = [e, g_1, g_2, \ldots, g_n]$ for $g_i \in B_2(e, \mathcal{S})$;
3. Solve the problem:

$$\begin{aligned} \text{maximize:} \quad & \lambda \\ \text{subject to:} \quad & P \succcurlyeq 0, \quad P \in \mathbb{M}_E(\mathbb{R}) \\ & \Delta^2 - \lambda\Delta = EPE^T \\ & \lambda \geq 0; \end{aligned}$$

4. Compute $Q = [q_1, \ldots, q_n] \sim \sqrt{P}$:

$$P \to \sqrt{P} \to \sqrt{P}_{\mathbb{Q}} \to \sqrt{P}_{\mathbb{Q}}^{aug} \to Q \in \mathbb{M}_E(RIF)$$

5. Setting $\xi_i = \langle E, q_i \rangle$ we have

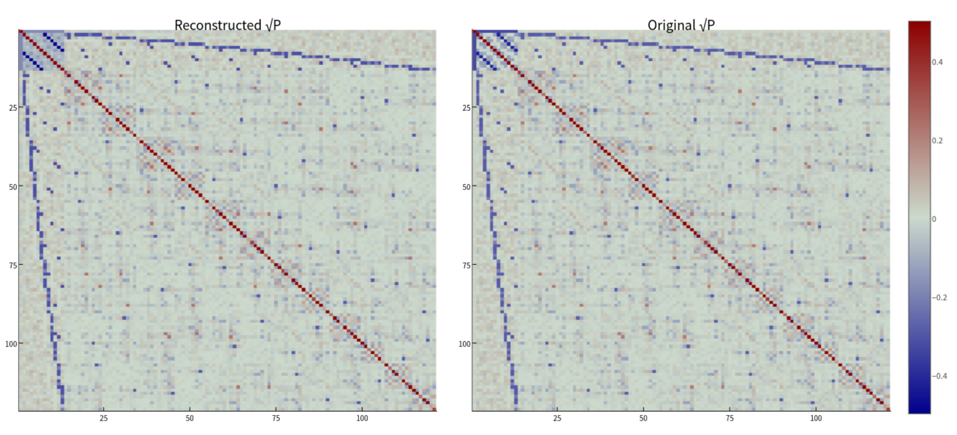
$$\Delta^2 - \lambda\Delta = \sum \xi_j{}^* \xi_j + r, \quad \text{where } r \in I[G] \text{ and } \|r\|_1 < \varepsilon.$$

6. Finally $\Delta^2 - (\lambda - 2^{d-1}\varepsilon)\Delta = \sum \xi_j^* \xi_j + \left(r + 2^{d-1}\varepsilon\Delta\right) \geq 0$, hence

$$\lambda_1(G, \mathcal{S}) \geq (\lambda - 2^{d-1}\varepsilon) \quad \text{is certified.}$$

Tables and pictures

Group $SL_3(\mathbb{Z})$



These belong to $\mathbb{M}_E(\mathbb{R})$, where $E = B_2(e, E(3))$, i.e. rows and columns are indexed by elements in $(\mathrm{SL}(3, \mathbb{Z}), E(3))$ of word length ≤ 2.

G	n	m	λ_1	$\|r\|_1 <$	lb_κ	$< \kappa$	ub_κ
$SL(3, \mathbb{Z})$	390,287	935,021	0.54050	$5.2 \cdot 10^{-7}$	0.19	0.30014	0.81650
$SL(4, \mathbb{Z})$	93,962	263,122	1.31500	$5.2 \cdot 10^{-8}$	0.00106	0.33103	0.70711
$SL(5, \mathbb{Z})$	628,882	1,757,466	2.65000	$2.0 \cdot 10^{-4}$	0.00105	0.36400	0.63246

Thank You

Bibliography

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