$\mathrm{SAut}(F_5)$ HAS PROPERTY (T)

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Kummamoto 2018

Outline

Group Laplacians & Property (T)

Positivity & SDP

The procedure

Concrete examples

Notation / Convention

- $G = \langle S \mid R \rangle$ is a finitely presented group generated by a fixed **symmetric generating set** (i.e. $S^{-1} = S$).

Group Laplacians

Group Laplacian Δ

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- spectrum of Δ is real and non-negative;
- the second eigenvalue λ_1 is called the **spectral gap**

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots$$

Property (T)

- For an orthogonal representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$ of G on a (real) Hilbert space $\mathcal{H}$ denote by

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$$\kappa(G, S) = \inf_{\pi} \kappa(G, S, \pi)$$

over all orthogonal representations π of G. We say that G has the **Kazhdan's property (T)** if and only if there exists a (finite) generating set S such that $\kappa(G, S) > 0$.

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- Does $\mathrm{SAut}(F_n)$ have property (T) for $n \geqslant 4$? (Serre 70s, Lubotzky 1994, Lubotzky-Pak 2001, Fisher 2006, Bridson-Vogtmann 2006, Breuillard 2014, and many more);

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- Provide a constructive (computable) proof for $n = 5$.

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Theorem (Lubotzky & Pak, 2000)

*Let K be a finite group generated by $k \leq n$ elements. If $\mathrm{SAut}(F_n)$ has property (T) with constant $\kappa = \kappa(\mathrm{SAut}(F_n), \{\text{transvections}\}) > 0$, then **PRA walk** on $\Gamma_n = \Gamma_n(K)$ has fast mixing rate, i.e.*

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$$\left\|Q^t_{(g)} - U\right\|_{\mathrm{tv}} \leqslant \varepsilon \qquad \text{for} \qquad t \geqslant \frac{16}{\kappa^2} \log \frac{|\Gamma_n|}{\varepsilon} \sim O\left(\left(\frac{n}{\kappa}\right)^2 \log \frac{|K|}{\varepsilon}\right)$$

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Note

We do observe fast mixing rate in practice for large n.

Property (T) and Δ

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Corollary

Let $G = \langle S | \ldots \rangle$ be a finitely generated group. If there exists $\lambda > 0$ such that $\Delta^2 - \lambda\Delta \geqslant 0$, then G has property (T) with

$$\sqrt{\frac{2\lambda}{|S|}} \leqslant \kappa(G, S).$$

How to prove that $\Delta^2 - \lambda\Delta \geqslant 0$?

How to prove that a polynomial $f \geqslant 0$?

Positivity & SDP

Theorem (Hilbert's Positivstellensatz, 1888)

An everywhere non-negative polynomial $p \in \Sigma^2\mathbb{R}[x_1, \ldots, x_n]$ *(is a sum of squares) if and only if either*

- $n = 1$, *or*
- $n = 2$ *and* $\deg p = 2, 4$.

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$$p \geqslant 0 \iff \exists q \colon q^2 p \in \Sigma^2\mathbb{R}[x_1, \ldots x_n]$$

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$x^4y^2 + x^2y^2 - 3x^2y^2 + 1 \geqslant 0$ is SOS if You multiply is by $x^2 + y^2 + 1$.

SOS decomposition

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By evaluating

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we can obtain any polynomial of degree 2 in x and y with coefficients linear functions of p_{ij}.

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For f to be a SOS we just need $(p_{ij}) = P$ to be **semi-positive definite**,

since then $P = Q^T Q$ and (for $(1,x,y)^T = \vec{\mathbf{x}}$)

$$f = \vec{\mathbf{x}}^T P \vec{\mathbf{x}} = \vec{\mathbf{x}}^T Q^T \cdot Q\vec{\mathbf{x}} = (Q\vec{\mathbf{x}})^T \cdot (Q\vec{\mathbf{x}})\,.$$

Mathematical Programming

Linear programming:

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Semi-definite programming

- optimise linear functional
- on a polytope intersected with the cone of SPD matrices (spectrahedron)
- weak duality, non-unique solutions
- even feasibility is a hard problem!

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tries to maximise λ as long as $(2x^2 + 4x + 1) - \lambda \geqslant 0$.

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Theorem (Schmüdgen)

For any $*$-invariant element $\xi \in \mathbb{R}[G]$

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This of no use for us: SOS decompositions $\Delta^2 - \lambda\Delta + \varepsilon = \sum \xi_i^* \xi_i$ may be very diffrent for different ε.

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If we can show that $\Delta^2 - \lambda\Delta + \varepsilon_0\Delta = \sum \xi_i^* \xi_i$ for a single fixed ε_0, then

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for all ε simultanuously!

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4. Compute $\sqrt{P} = Q = [\overrightarrow{q_e}, \ldots, \overrightarrow{q_{g_n}}]$
5. Finally: $\xi_g = \langle \vec{\mathbf{x}}, \overrightarrow{q_g}\rangle$ and $\Delta^2 - \lambda\Delta = \sum_{g\in\vec{\mathbf{x}}} \xi_g{}^* \xi_g$.

How do we certify that the numerical result is sound?

Certifying correctness

Lemma (Netzer&Thom)

Let $r \in I[G] \subset \mathbb{R}[G]$ *such that* $\operatorname{supp}(r) \subset B_d(e)$. *Then*

$$r + 2^{d-1}\|r\|_1 \cdot \Delta \in \Sigma^2 I[G].$$

Corollary

If $\Delta^2 - \lambda\Delta = \sum \xi_i^* \xi_i + r$, *then*

$$\Delta^2 - \left(\lambda - 2^{d-1}\|r\|_1\right)\Delta = \sum \xi_i^* \xi_i + \left(r + 2^{d-1}\|r\|_1 \Delta\right) \in \Sigma^2 I[G],$$

i.e. Δ *has spectral gap of at least* $\lambda - 2^{d-1}\|r\|_1$.

Action Plan 2

1. Pick $G = \langle S | \mathcal{R} \rangle$;
2. Set $\vec{\mathbf{x}} = (e, g_1, g_2, \ldots, g_n), \quad g_i \in B_d(e, S)$ $(d = 2, 3, \ldots)$;
3. Solve the problem (numerically):

$$
\begin{aligned}
\text{maximize:} \quad & \lambda \\
\text{subject to:} \quad & P \succcurlyeq 0, \quad P \in \mathbb{M}_{\vec{\mathbf{x}}} \\
& \lambda \geq 0 \\
& (\Delta^2 - \lambda\Delta)_t = \left(\vec{\mathbf{x}} P \vec{\mathbf{x}}^T\right)_t, \quad \text{for all } t \in B_{2d}(e, S)
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$$P \to \sqrt{P} \to \sqrt{P}_{int} \to \sqrt{P}_{int}^{aug} \to Q \in \mathbb{M}_{\vec{\mathbf{x}}}(\mathbb{R}IF)$$

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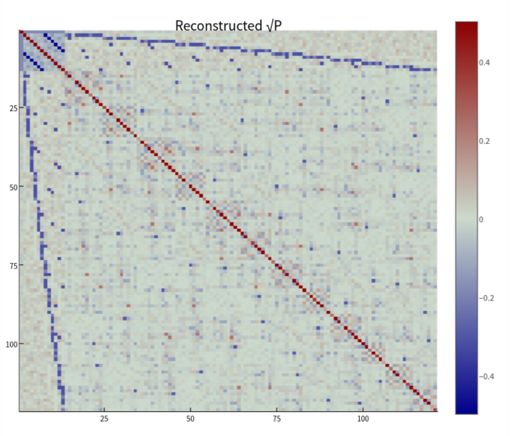
Concrete examples

$SL(3, \mathbb{Z})$



$\sqrt{P} = Q \in \mathbb{M}_{\vec{\mathbf{x}}}$, where $\vec{\mathbf{x}} = B_2(e, E(3))$, i.e. rows and columns are indexed by elements in $(\mathrm{SL}(3, \mathbb{Z}), E(3))$ of word length $\leqslant 2$. In this case

$$\Delta^2 - 0.28\Delta = \sum_{i=1}^{121} \xi_i^* \xi_i + r, \qquad \|r\|_1 \in [3.8508, 3.8511] \cdot 10^{-7}$$

SL(n, $\mathbb{Z}$)

G	n	m	λ	$\|r\|_1 <$	lb_κ	$< \kappa$	ub_κ
SL(3, $\mathbb{Z}$)	390,287	935,021	0.5405	$5.2 \cdot 10^{-7}$	0.19	0.30014	0.81650
SL(4, $\mathbb{Z}$)	93,962	263,122	1.3150	$5.2 \cdot 10^{-8}$	0.00106	0.33103	0.70711
SL(5, $\mathbb{Z}$)	628,882	1,757,466	2.6500	$2.0 \cdot 10^{-4}$	0.00105	0.36400	0.63246

G	n	m	λ	$\|r\|_1 <$
SAut(F_4)	3,157,730	1,777,542	0.0100	7.4

(after weeks of computation)

SAut(F_5)

G	n	m
SAut(F_5)	21,538,881	11,154,301

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- Find a finite group $K < \mathrm{Aut}(\mathrm{SAut}(F_n))$ which keeps the generating set S and (thus) $\Delta^2 - \lambda\Delta$ invariant ($K = \mathbb{Z}_2 \wr S_5$);

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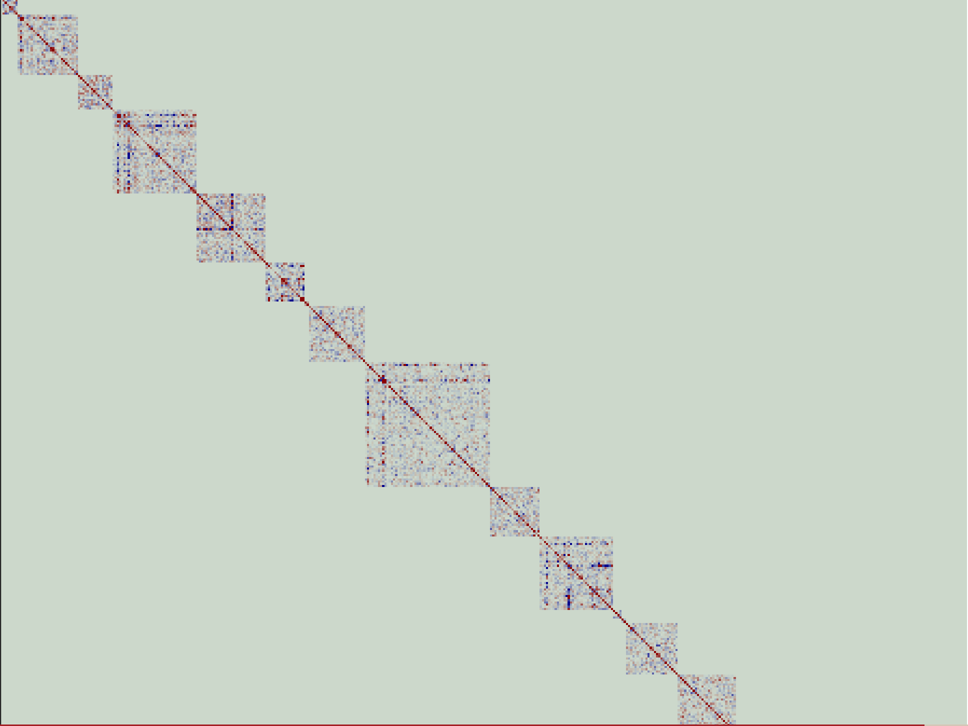
- Using minimal projection system for K reduce the size of the optimisation problem (29 semidefinite constraints, 13233 variables)
- Solve the smaller problem and reconstruct the solution P of the larger one.



SAut(F_5)

G	n	m	λ	$\|r\|_1 <$	$< \kappa$
SAut(F_5)	13,233	7,230	1.3000	$2.1 \cdot 10^{-6}$	0.18028

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(*The Method of Mechanical Theorems*, Archimedes)